Analysis of Generalised Boussinesq Coupled Equations Using Lie Symmetry

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Abstract- In the last decades, Nonlinear partial differential equations (NPDEs) have become essential tools to model complex phenomena that arise in different aspects of science and engineering such as hydrodynamics. Therefore, constructing exact and approximate solutions of NLPDEs is of great importance in mathematical sciences. Previously authors have done similar work with restriction of K and L to be one. In this paper we solve the generalised Boussinesq coupled equations:

$$ut + Kvx + Luux = 0;\ K > 0;\ L > 0$$
$$vt + uvx + uxxx = 0$$

using Lie symmetry of differential equations where u = u (x; t) is the velocity of water and v = v (x; t) the total depth of water and subscripts denote partial derivatives. The positive constants K, L would enable further analysis of optimal water depth and velocity be determined.

I. INTRODUCTION

In this paper, we apply Lie symmetry method to study the coupled Boussinesq system which consists of two non-linear Partial Differential Equations given by;

$$ut + Kvx + Luux = 0 \qquad 1(a)$$
$$vt + (uv)x + uxxx = 0 \qquad 1(b)$$

II. LIE GROUPS ADMITTED BY THE SYSTEM 1(A) AND (B)

The one-parameter Lie groups are generated by exponentiating the generators or integrating the Lie equations i.e.

$$(x\_;\ t\_;\ u\_;\ v\_) = e"Gi(x;\ t;\ u;\ v)$$

OR

$$\frac{dx^{*^i}}{d\varepsilon} = \xi^i(x^*, u^*) \qquad \frac{du^{*^\alpha}}{d\varepsilon} = \eta^\alpha(x^*, u^*)$$

with initial conditions

$$x^{*^i}|_{\varepsilon=0} = x^i \text{ and } u^{*^\alpha}|_{\varepsilon=0} = u^\alpha$$

Hence: For

$(i)\ \ X_1 = \frac{\partial}{\partial x} \qquad \frac{dx^*}{d\varepsilon} = 1, dx^* = d\varepsilon; x^* = \varepsilon + k$

$x^*|_{\varepsilon=0} = 0 + k = x, x^* = x + \varepsilon$

$(ii)\ \ X_2 = \frac{\partial}{\partial t}$ will be; $t^* = t + \varepsilon$

$(iii)\ \ X_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}$

$\frac{dx^*}{d\varepsilon} = t^*, dx^* = td\varepsilon; \qquad x^* = \varepsilon t + k$

$x^*|_{\varepsilon=0} = k = x, x^* = \varepsilon t + x$

$\frac{du^*}{d\varepsilon} = 1, \qquad u^* = \varepsilon + u$

$(iv)\ \ X_4 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} - 3v\frac{\partial}{\partial v}$

$\frac{dx^*}{d\varepsilon} = x^* \qquad \frac{dx^*}{x^*} = d\varepsilon; \qquad lnx^* = \varepsilon + k$

$x^* = e^{\varepsilon+k}, x^* = ke^{\varepsilon}$

$x*|_{\varepsilon=0} = k = x$; Hence $x^* = xe^{\varepsilon}$

$t^* = te^{\varepsilon}$

$\frac{du^*}{d\varepsilon} = -2u^*, \frac{du^*}{u^*} = -2d\varepsilon; lnu^* = -2\varepsilon + k$

$u^* = ke^{-2\varepsilon}, u^*|_{\varepsilon=0} = k = u$

Hence $u^* = ue^{-2\varepsilon}$ and $v^* = ve^{-3\varepsilon}$

The above symmetries can be summarised as follows:

$$X_1 = \frac{\partial}{\partial x} : F_1(x,t,u,v) = (x+\varepsilon, t, u, v) \quad 2(a)$$
$$X_2 = \frac{\partial}{\partial t} : F_2(x,t,u,v) = (x, t+\varepsilon, u, v) \quad 2(b)$$
$$X_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} : F_3(x,t,u,v) = (x+\varepsilon t, t, \varepsilon+u, v) \quad 2(c)$$
$$X_4 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} - 3v\frac{\partial}{\partial v} : F_4(x,t,u,v) = (xe^{\varepsilon}, te^{\varepsilon}, ue^{-2\varepsilon}, ve^{-3\varepsilon}) \quad 2(d)$$

Discussing the above groups give physical meanings.

X1 and X2 are translations along x and t respectively.

That is X1 and X2 are trivial groups which are translations. Their solutions are invariant.

So, from above it is clear that F3 is the only Lie Symmetry mapping. From F3 and its inverse mapping we immediately obtain a Lie Symmetry solution to the Coupled Boussinesq curve using [F3] inverse on the Invariant solution curves.

2.1 Transformation Groups of solutions of the system 1

$$u_1^* = \mu^\alpha(x-\varepsilon, t, u, v) \quad 3(a)$$
$$u_2^* = \mu^\alpha(x, t-\varepsilon, u, v) \quad 3(b)$$
$$u_3^* = \mu^\alpha(x-\varepsilon t, t, \varepsilon+u, v) \quad 3(c)$$
$$u_4^* = \mu^\alpha(xe^{-\varepsilon}, te^{-\varepsilon}, ue^{2\varepsilon}, ve^{3\varepsilon}) \quad 3(d)$$

2.2 Invariant Solutions of the System 1

Case 1

Given the generator $X = \partial/\partial x$
The characteristic equation is:

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0} = \frac{dv}{0}$$

$$dt = 0$$

$$t = c$$

$$u(x,t) = \Phi(t)$$

$$v(x,t) = \varphi(t)$$

If each group is a symmetry group and $u\alpha = \mu\alpha\ (x;\ t)$ is a solution of the system 1; then the functions $(u^\alpha) * \mu$
are also solutions for α = 1, 2, 3…

Given that (x*; t*; u*; v*) are group transformations and (u*) α = X (x; t; u; v); then by the inverse mapping theory [6], the symmetry solutions u*(x; t) satisfies the relation

$$(u^*)^\alpha = U\{g_\varepsilon^{-1}(x^*), g_\varepsilon^{-1}(t^*), [\mu^\alpha(g_\varepsilon^{-1})(x^*), g_\varepsilon^{-1}(t^*))]; \varepsilon^{-1}\}$$

where μ^α is any known solution of the system. Hence;

On integration we get

$$ut = c$$
$$vt = c$$

We substitute in the Boussinesq system to obtain:

$u\ (x,t)\ =\ c.(u+\varepsilon)$ 4(a)
$u\ (x,t)\ =\ c.e2\varepsilon$ 4(b)
$v\ (x,t)\ =\ c.e3\varepsilon$ 4(c)

Case 2

Given the generator $X = \partial/\partial t$
The characteristic equation is:

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0} = \frac{dv}{0}$$

on integration we obtain:

$$x = c, u = g(x), v = f(x)$$

$$u_t = 0, v_t = 0$$

$$u_{xxx} = g'''$$

$$v_x = f'$$

$$g' = \frac{dg}{dx}, f' = \frac{df}{dx}$$

We substitute into the Boussinesq system:

$$u_t + Kv_x + Luu_x = 0$$

$$v_t + (uv)_x + u_{xxx} = 0$$

$$Kf'(x) + Lg(x)g'(x) = 0$$

$$L\int gg' = -K\int f'$$

$$\int Lgg'dx = L\int gg'dx = z$$

$$\int z = uv - \int vdu$$

$$z = g^2 - \int gg'dx$$

$$z = g^2 - \frac{1}{L}Z$$

$$\frac{Z}{1} + \frac{Z}{L} = g^2$$

$$\frac{LZ + Z}{L} = g^2$$

$$Z(L+1) = Lg^2$$

$$Z = \frac{Lg^2}{L+1}$$

$$\int -Kf'(x) = -K\int f'(x)dx$$

$$-K\int \frac{df}{dx} = f'(x)$$

$$f = f(x) + K$$

$$-K(f+k) \Rightarrow - \Leftrightarrow -Kf + k_1$$

$$\frac{Lg^2}{L+1} = -Kf + k_1$$

$$Lg^2 = (L+1)(-Kf + k_1)$$

$$Lg^2 = -KLf(x) + Lk_1 - Kf + k_1$$

$$g^2 = -Kf + k_1 - \frac{K}{L}f + \frac{k_1}{L}$$

$$g^2 = -Kf - \frac{K}{L}f + k_2$$

$$g(x) = \sqrt{-Kf - \frac{K}{L}f + k_2}$$

$$\int \frac{df}{dx} = \int gf' = y$$

$$u = g(x), du = g', dv = f', v = f$$

$$-\int fg'dx$$

$$u = f, du = f', dv = g', v = g$$

$$fg - \int gf'$$

$$y = gf - [gf - y]$$

$$u(x,t) = \sqrt{(Kf - \frac{K}{L}f + k_2).e^{-2\varepsilon}} \quad 4(d)$$

$$u(x,t) = \sqrt{(Kf - \frac{K}{L}f + k_2) + \varepsilon} \quad 4(e)$$

$$v(x,t) = k_3 \int\int e^{g(x)}dxdx.e^{-3\varepsilon} \quad 4(f)$$

Using X_3 and X_4

$$u(x,t) = \mu_u.(u+\varepsilon) \quad 4(g)$$

$$u(x,t) = \mu_u.e^{2\varepsilon} \quad 4(h)$$

$$v(x,t) = \mu_v.e^{3\varepsilon} \quad 4(i)$$

Where μ is a known solution or invariant solution of the system 1. For each invariant solution μ; we can obtain the corresponding symmetry solution u^{α} (x; t) by inserting the invariant solution μ into the symmetry solution u^{α} (x; t) as shown.

$$u(x,t) = c.(u+\varepsilon) \quad 4(a)$$

$$u(x,t) = c.e^{2\varepsilon} \quad 4(b)$$

$$v(x,t) = c.e^{3\varepsilon} \quad 4(c)$$

CONCLUSION

In this paper, we have solved the generalised Boussinesq coupled equations:

$$ut + Kvx + Luux = 0,$$
$$vt + (uv)x + uxxx = 0$$

using Lie symmetry of differential equations. We have obtained groups admitted by the system. Exact solutions are obtained using inverse mapping theory. Transformation groups and invariant solutions have also been obtained.

RECOMMENDATIONS

We would like to recommend that in cases where some techniques have failed to solve certain differential equations, lie symmetry method can be applied since it does not depend on any restriction on parameter values.

For instance; using X1 and X2 to develop final solutions, we have;

REFERENCES

[1] Mohammed O. A. 2019. Solution of the coupled Boussinesq-Burger's equations by reduced differential transform method. ResearchGate.

[2] Shreekant P. P.,Twinkle S.2016.Solution of coupled non-linear system by optimal homotopy analysis method. ResearchGate.

[3] Anwar J. M. J.,Mahmood J.A.2018.Soliton Solutions of the Coupled Schrdinger-Boussinesq Equations for Kerr Law Nonlinearity. Hindawi.

[4] Hong-Qian S.,Ai-Hua C.2018.Exact solutions of the classical Boussinesq system. Taylor and Francis.

[5] Kamal R., Talaat S. EL-Danaf, Khalid A.2016.An efcient approach to numerical study of the coupled-bbm system with b-spline

collocation method. Communication in Mathematical Modeling and Applications. CMMA 1, No. 3, 5-15 (2016)

[6] Oliver P., J. Applications of Lie Groups to Differential Equations,2nd ed. (1993). Springer-Verlag, Berlin.