# A Note on Duality in the FK-Spaces

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***Abstract- We define a subspace $X_0 \subset X$ where X a semi conservative space, discus and derive some inclusion properties between the α−, γ− and f− duals of the subspace and the duals of space X. We show that for equality of the duals to hold, X must have the semi conservative property.***

## I. INTRODUCTION

Let w $C^N$ denote the set of all Real or complex valued sequences where C is the set of all complex numbers and N the set of all-natural numbers including 0. A linear subspace of w is called a sequence space.

- Examples 1. $l_\infty$, c, and $c_0$ be the linear spaces of bounded, convergent and null sequences x = $(x_k)$ with complex terms, respectively, normed by

$$\|x\|_\infty = \sup_k |x_k|$$

where k ∈ N = {1, 2...}, the set of positive integers. $K_1$zmaz [4] defined the sequence spaces
$l_\infty$ (4) = {x = $(x_k)$: 4x ∈ $l_\infty$},
c (4) = {x = $(x_k)$: 4x ∈ c}, $c_0$(4) = {x = $(x_k)$: 4x ∈ $c_0$}
where 4x = $(4x_k)$ = $(x_k - x_{k+1})$, and showed that these are Banach spaces with norm $\|x\| = |x_1| + \|4x\|_\infty$.

After then Et [1] defined the sequence spaces $l_\infty$ $(4^2)$ = {x = $(x_k)$: $4^{2X}$ ∈ $l_\infty$}, c $(4^2)$ = {x = $(x_k)$: $4^2$x ∈ c}, $c_{0(4}2_{)}$ = {x = $(x_k)$: $4^2$x ∈ $c_0$}

- Definition 2. A coordinate space K, is a linear space X of sequences with a locally convex topology for which the inclusion map is continuous. Addition and multiplication are done coordinate wise. An FK space is a complete metrisable locally convex topological vector space consisting of complex sequences and with the property that the coordinate linear functionals are continuous. An FK space whose topology is normable is called a BK space
- Definition 3. An FK space X is said to have the AK property if X ⊃ ω and AD property if ω is dense in X has the AB property when sup $X^n < \infty$ where $X^n = (x_0,x_1,x_2,\ldots 0,0,\ldots)$
- Definition 4. A space X is o−conservative if X ⊃ $c_0$ and semiconservative if ${}^p\delta^k$is weakly Cauchy. i.e. ${}^p f(\delta^k)$ is convergent for all f ∈ $X^0$,

The application of FK spaces in the theory of Matrix transformations stems from the fact that Matrix mappings between FK spaces are continuous. For example, in economics, Hilbert spaces U may represent commodity spaces [or functions with values in commodity spaces]. An element f of the dual associates to every 'commodity' x = U a scalar [f,x] = R which when we interpret this scalar as the value of the commodity x, f can then be considered as a 'price' , the role of which is to associate a value to every commodity. therefore, the duality theory has proved very interesting for mathematicians. Various properties and characterizations have been done on the dual spaces.

In our paper we look at the relation between the duals of a space X as they relate to a very special space, the $X_0$ = Kerf. First, we shall establish the relations that exist between the α−, β−, γ− and x− duals for the same space X.

- Definition 5. Let X be a sequence space. Then we define

  1. $X^\alpha$ = {a ∈ ω : ${}^P{}_k |a_k x_k| < \infty$, for all x ∈ X,
     :⇒ ∩x∈X(x−1∗l1)
  2. $X^\beta$ = a = $(a_k)$ : ${}^P{}_k\, a_k x_k$ is convergent, for all x ∈ X},
     :⇒ $\cap_{x\in X}(x^{-1} * cs)$
  3. $X^\gamma = \{a \in \omega : \sup_n |\sum_{k=1}^n a_k x_k| < \infty$, for all x ∈ X}

$:\Rightarrow \cap_x \in_X (x^{-1} * bs)$

Then $X^\alpha, X^\beta, X^\gamma$ are called α−,β−,γ - dual spaces of X, respectively. The duals are also referred to as the Kothe-Toeplilitz and bounded dual respectively. It is easy to show that $\varphi \subset X^\alpha \subset X^\beta \subset X^\gamma$. It has also been shown that if

If $X \subset Y_-$ then $Y^\eta \subset X^\eta$ for $\eta = \alpha,\beta,\gamma$.

- Definition 6. The functional dual (or the f−dual) is defined as

$X^f :\Rightarrow \{(g(\delta^k) : g \in X\}$

We shall define a space and discus these duals as they relate to the subspace. The concept of sequence spaces have been studied by various mathematicians,

## II. TERMINOLOGY AND PRELIMINARIES

Most terminology is standard as by Wilansky and Kamthan, but we mention a few for ease of reading. ω shall mean the set of all finite sequences.3 Some few results Throughout the paper we write $\Sigma_k$ for $\sum_{k=1}^{\infty}$ and $\lim_n$ for $\lim_{n ß \infty}$.

- Lemma 7. ([4]). Let $(p_n)$ be a sequence of positive numbers increasing monotonically

to infinity. i) If $\sup_n |\sum_{v=1}^{n} p_v a_v| < \infty$, then $\sup_n |p_n \sum_{k=n+1}^{\infty} a_k| < \infty,\ ii)\ If \Sigma_k p_k a_k$ is convergent, then $\lim_n p_n \sum_{k=n+1}^{\infty} a_k = 0$.

- Lemma 8. ([2]). $x \in l_\infty(4^m)$ implies $\sup_k k^{-m}|x_k| < \infty$.

- Theorem 9. Let X be an FK− space such that X = [E] Then X = [Ecl] , [Ecl] is the closure of E $._0$

Proof. X is the sequence of the form sup [X]X = X Let u = X say u = f[x].Let g be the restriction of f to [E] then U = g[X][....] u = [E] And so X = [E cl] Conversely;

Let u = [Ecl] say u = g[x] By the Hahn-Banach theorem extend g to f = X Then U = f[g] and hence u = X. Hence [Ecl] = X and the result fpllows.

- Theorem 10. If $X \subset Y$ then $X^f \supset Y^f$. If X is closed in Y , Then $X^f = Y^f$

Proof. If $f \in Y^*$, Let $u \in Y^f$ Say $u_n = f(\delta^n)$ Then,

Let g be the restriction of f to X, then $g \in X^*$ hence $u_n = g(\delta^n) \Rightarrow u \in X^f$.

Hence $Y^f = X^f$

If $X \subset Y$ then the X topology is larger than the Y topology and they are equal only if X is closed in Y , so that $X^f = Y$

- Theorem 11; Let X be an FK space containing $E^\infty$, Then

i. $X^\beta \subset X^f$
ii. If X has AK then $X^\beta = X^f$

Proof;

i) Let $u \in X^\beta$ and define $f(x)] = \sum_{n=1}^{\infty} u_n x_n$ for $x \in X$. Then $f \in X^*$ by the

Banach-steinhaus theorem. Also $f(\delta^n) = u_n$ so $u \in X^f$. Hence $X^\beta \subset X^f$ (1)

ii) Let $u \in X^f$ say $u_n = f(\delta^n)$. For $x \in X$ $f(x) = f(\sum_{n=1}^{\infty} x_n \delta^n)$ since X has AK

= P∞n=1 $x_n(\delta n)$

= P∞n=1 $x_n u_n \Rightarrow u \in X^\beta$

Hence $X^f \subset X^\beta$ (2)

From (1) and (2) the result follows.

- Theorem 12; If X is an AK space then $X^* \subset X^\beta$

Proof. Let $f \in X^*$ and $x \in X$ be arbitrary. Then x = P∞k=1
xkδk. $f(x) = {}^{P\infty}{}_{k=1}\, x_k(\delta^k)$ Hence $f(\delta^k) \in X^\beta$. And so $X* \subset X\beta$

- Theorem 13; If X is a barreled AK space then $X^\beta = X^*$

Proof ; Let $u \in X^\beta$ and define $f_n = X^*$ by $f_n(X) = {}^P{}_k{}^n{}_{=1}\, x_k u_k$ Since $f_n$ is point wise bounded it follows from the Banach-Steinhaus theorem that $f \in X**$ where

$$f(x) = \sum_{k=1}^{\infty} x_k u_k$$

But $f \in (\delta^n)_0^\infty$ and so $(u)_0^\infty = (\delta k_0^\infty) \in X^*$

Hence u = X* and therefore $X^{\beta} \subset X^*$. The converse has already been proved above for a general AK space. Hence

$X^{\beta} = X^*$

## III. MAIN RESULTS

We define a subspace $X_o \subset X$ such that $X_0$ = Kerf where f : X → Y ; X, Y normed linear spaces. We aim to show that $X_0^{\zeta} = X_{\zeta}$, where ζ ∈ {α, γ, f}. Remark Let X be a BK− space with $\Delta^+ = \{\delta, \delta^0, \delta^1 \dots\}$ as its Schauder basis, then every x = X is of the form

$$x = \lambda\delta + \sum_{k=0}^{\infty} \lambda_k \delta^k \text{ .................}(1)$$

Where $\{\lambda_0,\lambda_1,....\}$ is a sequence of scalars in C. (1) is equivalent to

$$x = \lambda\delta + \sum_{k=0}^{\infty}(x_k - \lambda)\delta^k \text{ ...............}(2)$$

If we define P → C where X is as in (2) and let

Ker P = $P^{-1}\{0\}$ = {x ∈ X : P(x) =0} = $X_0$, where 0 is the zero sequence

Then,

- Theorem 14

1. P is a continuous linear functional
2. $X_0$ is a BK space with AK in the X−topology

Proof. 1. By the hypothesis X is a BK space and so all co-ordinate functionals are continuous with respect to any basis, hence p = X*.

2. Since P is continuous, $P^{-1}\{0\}$ is a closed subspace of X. But P⁻1{0} = KerP = $X_0$. Now every x = X has the representation $x = \sum_{k=1}^{\infty} x_k \delta^k$ In the x topology, therefore x is a BK space with AK.

Furthermore, if X above semi-conservative then we have the following results

- Proposition 15. $X^{\alpha} = X_0^{\alpha}$

Proof; Let t = $X^{\alpha}$ then $x = \sum_{k=0}^{\infty} \mid x_k t_k \mid < \infty$ for all t = X and in particular for t = $X_0$ since $X_0 \subset X$ Hence $t = X_0^{\alpha}$. And thus $X \subset X_0^{\alpha}$ .......... [2a]

Conversely If $t = X_0^{\alpha}$ and x ∈ X is given by (2) i.e.

$$x = \lambda\delta + \sum_{k=0}^{\infty}(x_k - \lambda)\delta^k$$

Then

xλδ ∈= $X_0$ and so

$$x = \sum_{k=0}^{\infty} \mid (x_k - \lambda)t_k \mid < \infty$$

Define

$$g : X \to C \text{ by } g(x) = \sum_{k=0}^{\infty} \mid (x_k - \lambda)t_k \mid \text{...............}(3)$$

Then

$$g_n(x) = \sum_{k=0}^{\infty} \mid (x_k - \lambda)t_k \mid \text{..........}(4)$$

is a continuous linear functional on X and by letting n → ∞ in (4), g ∈ X* by the Banach-Steinhaus theorem.

Clearly

$$g(\delta^k) = t_k \Rightarrow \sum_{k=0}^{\infty} \mid g(\delta_k) \mid = \sum_{k=0}^{\infty} \mid t_k \mid < \infty$$

which exists since X is semi conservative. Since $\sum_{k=0}^{\infty} \mid (x_k - \lambda)t_k \mid$ exists it then follows that $\sum_{k=0}^{\infty} \mid (x_k k \mid$ and hence

t ∈ $X^{\alpha}$

So that

$$X_0^{\alpha} \subset X^{\alpha} \text{........................}(5)$$

By (2a) and (5)

$$X_0^{\alpha} = X^{\alpha}$$

- Proposition 16; $X_0^{\gamma} = X^{\gamma}$

Proof. Given

$X_0 \subset X$ $X^{\gamma} = \{x \in s : \sup_n \mid \sum_{k=0}^{n} x_k y_k \mid < \infty$, for all, y ∈ X}

Since $X_0 \subset X$ this implies

$$y \in X_0 \Rightarrow X^{\gamma} \subset X_0^{\gamma} \text{.............[1]}$$

conversely

let $t \in X_0^{\gamma}$ and x ∈ X be given by

$x = \lambda\delta + \sum_{k=0}^{\infty}(x_k - \lambda)\delta^k$................(2) then

x − λδ ∈ $X_0$ and so $\sum_{k=0}^{\infty}(x_k - \lambda)t_k$ is convergent.

Define g : X → C by g(x) =

$$\sum_{k=0}^{\infty}(x_k - \lambda)t_k \text{.......................}(3)$$

Then

$g_n(x) = \sum_{k=0}^{\infty}(x_k - \lambda)\delta^k$ is a continuous linear functional on X and by letting n → ∞ in (3), g ∈ X* by the Banach-steinhaus theorem. Now

$\sup_{n\geq 0} \mid g(x) \mid = \sup_n \mid \sum_{k=0}^{n} \mid (x_k - \lambda)t_k$

since by letting $n \to \infty,\ g_n(x) = \mid \sum_{k=0}^{n} \mid (x_k - \lambda)t_k$ tends to $g(x) = \sum_{k=0}^{n}(x_k - \lambda)t_k$ and $\sum_{k=0}^{\infty} t_k = \sum_{k=0}^{\infty} g(\delta^k) < \infty$ since X is semi-conservative, hence t = $X^\gamma$ and so

$X_0^\gamma \subset X^\gamma$ ..................$(4)$.

By (1) and (4) we get

$X_0^\gamma = X^\gamma$

- Proposition 17. $X_0^f = X^f$

Proof. ; Let u = $X^f$ say $u_n = f(\delta^n)$. Let Let g be the restriction of f to $X_0$

Then

$u_n = g(\delta^n) or u \in X_0^f$

Hence $X^f \subset X_0^f$..................(1)

Conversely,

If $u \in X_0^f$ say $u_k = g(\delta^k$ and x ∈ X is given by

$x = \lambda\delta + \sum_{k=0}^{\infty}(x_k - \lambda)\delta^k$

Then $x - \lambda\delta \in X_0$

and so $\sum_{k=0}^{\infty}(x_k - \lambda)u_k$ is convergent.

Now

Define $g : X \to C$ by $g(x) = \sum_{k=0}^{\infty}(x_k - \lambda)\delta^k$,

Then

$g_n = \sum_{k=0}^{\infty}(x_k - \lambda)u_k$..............(2) is continuous linear functional on X and taking limits as $n \longrightarrow \infty$ in (2), g = $X^*$ by the Banach-Steinhaus theorem. $g(\delta^k) = u_k \Rightarrow \sum_{k=0}^{\infty} g(\delta^k) = \sum_{k=0}^{\infty} u_k < \infty$ since X is semi-conservative.

Hence $u \in X^f$

and $X_0^f \subset X^f$........................$(3)$

By (1) and (3) we have $X_0^f = X^f$

- Remark 18; This result does not hold if x is not semi-conservative. We show this by giving a few examples.
- Example 1. $l_1$ is a BK space with AK property. bv is not semi-conservative and $l \subset bv$.Their duals are as follows $l_1^\alpha = l_\infty \neq (bv)^\alpha = l_1$

$l_1^\gamma = l_\infty \neq (bv)^\gamma = bs$ Hence l = bv and l = bv

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