# Estimation of Volatility Using European Logistic-Type Brownian Motion When Asset Price is Discontinuous

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***Abstract- Volatility is a measure of how unsure we are about the future of asset price; hence its estimation is very important for implementation, valuation and derivative pricing of assets. Volatility forecast is crucial as it affects investment choice and is the key input to valuation of corporate and public liabilities. It gives the idea about the stability of stock prices. Relatively high volatility implies that the stock price varies continuously within relatively large interval. Volatility is the standard deviation of the continuously compounded rate of return of the stock, per year. Volatility forecast is also the most important parameter affecting prices of market-listed options of which trading volume increased in the last decade. Volatility of an asset as used by Black-Scholes model is assumed to be constant throughout the duration of the derivative. This study involves European logistic-type option pricing with jump diffusion. Using the knowledge of logistic Brownian motion with the aid of Dupire approach we develop a logistic Brownian motion with jump diffusion model for price process.***

***Indexed Terms- Black-Scholes formula, Jump diffusion, Logistic Brownian motion, Volatility, Wiener process.***

## I. INTRODUCTION

Stock prices may change due to the general economic factors such as demand and supply, government changing interest rates policy, change in inflation rates, the corporate tax rate increase, changes in economic outlook and capitalization rates. These bring about small or marginal movements in stock's price hence modeled by a Geometric Brownian motion. On the other hand, the stock's price may fluctuate due to announcement of some important information causing over-reaction or under-reaction of the asset prices due to good and/or bad news. These include; a formidable competitor entering the market, research and development expert leaving the industry, the company losing a big contract, a company making a breakthrough in its manufacturing process and an outbreak of infectious disease like the novel Covid-19. This information may emanate from the firm or industry. Such information that arrives at discrete points in time can only be modeled by a jump process. Thus, modeling stock prices is about modeling new information about the stocks. In the process of modeling stock prices it should be noted that stock price dynamics are reflected by movement of their values in uncertain way. According to Market Efficient Hypothesis (MEH), the past history of stock is assumed to be fully reflected in the present prices.

Traders often pay much attention to recent trends in returns. They believe that if a stock showed high returns recently, after some positive information about a company appeared, it is very likely to continue providing high returns. As a result, the market in general overreacts after announcement of good news Cutler [1991]. But traders that pay attention to fundamental values of a stock find stocks that are overpriced this way and sell them, thus dropping the price. This brings about mean reversion patterns. The larger magnitudes of prices fluctuations due to market overreaction causes misallocation of funds (the companies that have better investment opportunities may face lower share price and will collect less money from stock market than those with worse investment opportunities). Thus, it is a reason for inefficiencies in a stock market Engel [1991].

Geometric Brownian Motion has been an ongoing study in financial literature Black-Scholes [1973] and Merton [1975]. Its extension has been studied by many researchers and one of its kind is European-Logistic type option pricing model also known as Logistic Geometric Brownian motion Onyango [2003], Nyakinda [2011] and Oduor [2012] and among others. The use of jump diffusion process has been used in Geometric Brownian motion Apaka [2015] and

Bayraktar [2010] among others but not in European-Logistic type option pricing model.

## II. PRELIMINARIES

In this section, we discuss some fundamental concepts that will be of importance to our study:

### 2.1 Stochastic process

Any variable whose value changes over time in uncertain way is said to follow a stochastic process. Hence it obeys laws of probability. Mathematically, a stochastic process $X = [X(t); t \in (0, \alpha)]$ is a collection of random variables such that for each $t$ in the index set $(0, \alpha)$, $X(t)$ is a random variable where $X(t)$ is the state of the process at time $t$. A discrete time stochastic is the one where the value of the variable can only change at a certain fixed point in time. On the other hand, continuous time stochastic, change can take any value within a certain range. In mathematical modeling stochastic process contains two parts. The first part is deterministic. This is the expected term which the dominant action of the system is modeled. The second part is the stochastic which represents the randomness along the dominant curve. Stochastic processes play a vital role in the mathematical treatment of financial instruments such as equities, commodities and derivatives contracts based on these Omollo [2010]. These processes can be used to model price not only traded products subject to random movements of markets, but also fixed-income products such as bonds, options and futures. The Brownian motion of market prices, also known in financial mathematics as the Wiener-Bachelier process can be traced back to the start of 20-th Century when a French mathematician, Bachelier [1964] presented his PhD thesis in which he analysed stock market fluctuations.

### 2.2 Markov Process

This is a particular type of Stochastic process where only the present value of the variable is relevant for predicting the future. It is believed that the current price already contains what is relevant from the past. It implies that the probability distribution of the price at any particular future time is not dependent on the particular path followed by the price in the past Hull [2005]. Stock prices are assumed to follow Markov process. Thus, the stock price fluctuations have the same probability distribution and independent of each other. This is in accordance to the random walk theory which states that the past movement or direction of the price of stock or overall market cannot be used to predict its future movement.

### 2.3 Wiener process or Brownian motion

It is a particular type of Markov Stochastic process with a mean change of zero and a variance of 1.0 per year. It follows a stochastic process where $\mu$ is the mean of the probability distribution and $\sigma$ is the standard deviation. That is W(t) ~ N (μ, σ) then for Wiener Process W(t) ~ N (0,1) which means W(t) is a normal distribution with $\mu = 0$ and $\sigma = 1$. If a variable Z follows a Wiener process then it has the following properties;

i. The change $\Delta Z$ for any two different short time intervals of time $\Delta Z = \epsilon\sqrt{\Delta t}$, where $\varepsilon$ has a standardized normal distribution; $\phi(0,1)$.
ii. The values of $\Delta Z$ for any two different short time intervals of time $\Delta t$, are independent that is $Var(\Delta Z_i, \Delta Z_j) = 0 \; i \neq j$ it follows that from the first property that itself has a normal distribution with mean of $\Delta Z = 0$ and standard deviation of $\Delta Z = \sqrt{\Delta t}$ and variance $\Delta Z = \Delta t$ that is $\Delta Z \to N(0, \sqrt{\Delta t})$ the second property implies that $Z$ follows a Markov process.

### 2.4 Generalised Wiener process

The basic Wiener process $dZ$ that has been developed so far has a drift rate of zero and a variance rate of 1.0. Here the drift rate of zero means that the expected value of $Z$ at any future time is equal to its current value. The variance rate of 1.0 means that the variance of the change in $Z$ in time interval of length $T$ equals T. A generalised Wiener process for a variable $X$ can be defined in terms of $dZ$ as

$$dX = adt + bdZ \tag{1}$$

where mean rate $a$ and variance rate $b$ are constants, $adt$ is the expectation of $dX$ and $bdZ$ is the addition of noise to the path followed by $X$, while $b$ is the diffusivity. In a small interval $\Delta t$, the change in the value of $X$, $\Delta X$ is of the form

$$\Delta X = a\Delta t + b\epsilon\Delta t \tag{2}$$

Whereas already defined above $\epsilon$ is a random variable drawing from standardized normal distribution thus the distribution of $\Delta X$ is $Mean\ = E(\Delta X) = a\Delta t$, variance $(\Delta X) = b^2\Delta t$ thus, standard deviation of $\Delta X$ = $b\ \Delta t$. Hence $\Delta X \sim N(a\Delta t, b\sqrt{\Delta t})$. Similar argument to those given for a Wiener process show that the change in the value of $X$ in any time interval $T$ is normally distributed with mean of change in $X = aT$ Standard deviation of change in $X = bT$, Variance of change in $X = b^2T$ Hence $dX \sim N(aT, bT)$

## 2.5 Itô's Process

Brownian motion is continuous everywhere but integrable nowhere. For this reason, ordinary rules of calculus cannot be performed on the stochastic component. To overcome this problem Kinyoshi Itô [1951] went on to develop stochastic calculus. This has become an essential tool that is used in Brownian to successfully model stock prices. Itô's Process is the generalised

Wiener process in which the parameters a and b are functions of the value of the underlying variable X and time t. An Itô's process can be written mathematicacally as;

$$dX = a(X,t)dt + b(X,t)dZ \qquad (3)$$

Both the expected drift rate and variance rate of an Itô's process is liable to change over time. In a small time, interval between $t$ and $t + \Delta t$, the changes from $X$ to $X + \Delta X$, is expressed as

$$\Delta X = a(X,t)\Delta t + b(X,t)\epsilon\sqrt{\Delta t} \qquad (4)$$

This relationship involves a small approximation. It assumes that the drift and variance rate of $X$ remain constant, equal to $a(X,t)\Delta t$ and $b^2(X,t)\Delta t$ respectively during the interval between $t$ and $t + \Delta t$ hence $\Delta X \sim N(a(X,t)\Delta t, b(X,t)\sqrt{\Delta t})$.

## 2.6 Itô's Lemma and its derivation

This is the formula used for solving stochastic differential equations. Suppose that the value of a variable X follows Itô's Process

$$dX = a(X,t)dt + b(X,t)dZ, \qquad (5)$$

where $dZ$ is a wiener process and $a$ and $b$ are functions of $X$ and $t$. The variable $X$ has a drift rate of $a$ and a variance of $b^2$. Itô's Lemma shows that a function $G(X,t)$ twice differentiable in $X$ and once in $t$, is also an Itô's process given by

$$dG = \left(\frac{\partial G}{\partial X}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}b^2\right)dt + \frac{\partial G}{\partial X}bdZ, \qquad (6)$$

Where the $dZ$ is the same Wiener process, thus $G$ also follows an Itô's Process with a drift rate of $\frac{\partial G}{\partial X}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}b^2$ and a variance rate of $\left(\frac{\partial G}{\partial X}\right)^2 b^2$. Equation (6) is the Itô's lemma.

## 2.7 Local volatility equation of dupire

The local volatility model was introduced by Dupire and Derman [1994]. This has become one of the most extensively used models in pricing of derivatives across asset classes. The Dupire equation enables us to determine the volatility function in a local volatility model from quoted call and put options in the market.

For a given current stock price $S(0)$ and a given expiration period T, the collection $C(S(0), K, T); K \in (0, \infty)$ of discounted option prices of different strikes yields the risk-neutral function $\phi$ of the final spot price $S(T)$

The price dynamics in the local volatility model under the risk neutral measure are given by;

$$dS(t) = (r(t) - q(t)S(t))dt + \sigma(S,t)S(t)dZ(t), \qquad (7)$$

Where $r(t)$ is the risk-free interest rate, $q(t)$ is a continuous dividend yield at time $t$, $\sigma(S,t)$ is the volatility and $Z(t)$ is the Wienner process. Then

$$C(S(0), K, T) = \int_K^\infty \phi(S(T), T; S(0))(S(T) - K)dS(T) \qquad (8)$$

Where $\phi(S(T), T; S(0))$ is the probability density of the final spot price at time T. This pseudo probability density function evolves according to Fokker-Plank equation

$$\frac{\partial \phi}{\partial T} = -S\frac{\partial(S(T)\phi)}{\partial T} + \frac{1}{2}\frac{\partial^2(\sigma^2(S(T))^2\phi)}{\partial S(T)^2} \qquad (9)$$

Differentiating equation (8) twice with respect to K gives

$$\frac{\partial^2 C}{\partial K^2} = \phi(S(T), T; S(0)) \qquad (10)$$

Differentiating equation (8) twice with respect to T we obtain;

$$\frac{\partial C}{\partial T}=\int_K^\infty[\frac{1}{2}\frac{\partial^2}{\partial S(T)^2}(\sigma^2 S(T)^2\phi)-\frac{\partial(\mu S(T)\phi)}{\partial T}](S(T)-K)dS(T) \quad (11)$$

Integrating (11) by parts twice gives

$$\frac{\partial C}{\partial T}=\frac{1}{2}\frac{\sigma^2K^2\partial^2 C}{\partial K^2}+\mu(T)C-K\frac{\partial C}{\partial K} \quad (12)$$

Equation (12) is called the Dupire local volatility equation

## III. MAIN RESULTS

We begin with deriving logistic Brownian motion also known as Non-linear Brownian Motion.

### 3.1 Logistic Brownian Motion- (Non-Linear Brownian Motion)

Most of the modern models have been modified to represent a non-linear variation of the famous Black-Scholes equation. Non-linear Black-Scholes equation tends to provide a better tool for predicting price changes by taking into account more realistic assumptions than that of the original Black-Scholes. This equation takes care of the transaction costs, illiquid markets, risks from unprotected portfolio and large investors preferences. These assumptions have a great impact on the stock price, the option price, volatility and the asset's growth rate Nyakinda [2018]. We obtain Logistic Brownian motion by introducing excess demand functions in the framework of the Walrasian (Walrasian-Samuelson) price adjustment mechanism Onyango [2003]. The asset price changes are directly driven by excess demand for a security. This is the core principle of Standard Walrasian model. To simplify the work, we do not allow cross-security effects that might be experienced when the market is multi-security market for which price of one security reacts to the excess demand of another. The dynamic adjustment rule in such simplified markets may be expressed in continuous-time Walrasian-Samuelson form by a rate of return;

$$\frac{1}{S(t)}\frac{dS(t)}{dt}=kED(S(t)), \quad (13)$$

where the parameters $t$ represents continuous time, and $k>0$ is a positive market adjustment coefficient (known as speed of market adjustment). $kED(S(t))$ is excess demand taken as continuous function of price $S(t)$. In terms of supply and demand functions $Q_S S(t)$ and $Q_D S(t)$, the excess demand is given by

$$ED\big(S(t)\big)=Q_D S(t)\text{-}Q_S S(t) \quad (14)$$

Applying the Walrasian-Samuelson model a logistic Brownian motion model or logistic stochastic differential equation is obtained which is;

$$\frac{dS(t)}{S(t)(S^*-S(t))}=\mu dt+\sigma dZ \quad (15)$$

Or

$$dS(t)=\mu S(t)\big(S^*-S(t)\big)dt+\sigma S(t)(S^*-S(t))dZ\ , \quad (16)$$

Where $S(t)$ is the price of the underlying asset at any time $t$, $S^*$ is the market equilibrium, $\mu$ is the rate of increase of the asset price $\sigma$ is the volatility of the underlying asset and $dZ$ is the Wienner process. Using Itô′s Lemma we can solve for $S(t)$ as

$$S(t)=\frac{S^*S(0)}{S(0)+(S^*-S(0))\exp\left(-(\mu S^*(t-t_0)+\sigma S^*Z(t)\right)} \quad (17)$$

This price dynamic is referred to us as logistic Brownian motion of stock price $S(t)$. Oduor [2012]

### 3.2 Estimation of Volatility using European Logistic-type Brownian Motion with jump diffusions

Walrasian price-adjustment model built a non-linear Brownian motion by introducing excess demand. A deterministic logistic equation is obtained by applying excess demand in the framework of Walrasian-Samuelson price adjustment mechanisms. Using the approach of Dupire we derive a diffusion process when the price follows non-linear Brownian motion.

Suppose that the price of the asset evolves according to the logistic jump diffusion equation

$$dS(t)=(\mu-\lambda k)S(t)\big((S^*-S(t)\big)dt+\sigma S(t)\big((S^*-S(t)\big)dZ+S(t)(S^*-S(t)(q-1)dN \quad (18)$$

Where $\mu$ is the growth rate, $\sigma$ is the volatility,$\lambda$ is the rate at which the jumps happen, $k$ is the average jump size measured as a proportional increase in asset price $S^*$ is the equilibrium price which is greater than asset price $S(t)$, $q$ is the absolute price jump size and $N$ is the Poisson process generating jumps. The aim is to show that there is a unique volatility function $\sigma(S,t)$ such that the observed option price is consistent with equation (18).

If we apply the black-Scholes Merton PDE for any claim of asset value $f(S,t)$, we have

$$\frac{\partial f}{\partial t}+\frac{\phi^2 s^2}{2}\frac{\partial^2 f}{\partial s^2}+\lambda E[f(qs,t)-f(s,t)]-\frac{\partial f}{\partial s}S\phi E(q-1)+r\phi s\frac{\partial f}{\partial s}-rf=0 \qquad (19)$$

Where $r(t)$ is the risk-free interest rate in the market since we are dealing with price derivatives?

If we consider a European call option the process or finding a fair option value of $f(S,t)$, will depend on asset price $S(t)$ and time $t$. Therefore, the function $f(S,t)$ can be written for the value of the contract with boundary condition

$$f(S,t)=\max(S^*-S(t),0) \qquad (20)$$

At a time $t$ before expiry date the price of the call option will be a function of $S(t),t,T$,and $S^*$ that is

$f((S(t)t,T,S^*)$ . when we fix the expiry rate T and the equilibrium $S^*$ we have

$$f(S,t)=\int_S^{\infty}\max(S^*-S(t)\phi dS(T) \qquad (21)$$

Differentiating (21) twice with respect to S we get

$$\phi=S=\frac{\partial^2 f(S,t)}{\partial K^2} \qquad (22)$$

Applying the Fokker-Plank equation and using Kolmogorov's foward equation on (18) we obtain

$$\frac{\partial \phi}{\partial T}-\frac{1}{2}\frac{\partial^2 f(S,t)}{\partial S(T)^2}\sigma^2 s^2\varphi^2\phi+\lambda E[f(qs,t)-f(s,t)]\phi-\frac{\partial f}{\partial s}S\varphi\phi E(q-1)+\frac{\partial f(S,t)}{\partial S(T)}r\varphi S\phi=0,$$

$$\text{where } \varphi=(S^*-S(t) \qquad (23)$$

Using the approach of Dupire [1994] taking f as a function of strike price in equation (23),

with differentiation taken with respect to drift and volatility function evaluated at S (because the density function in equation (22) is expressed in terms of S. Equation (23) can be re-written as;

$$\frac{\partial \phi(S)}{\partial T}-\frac{1}{2}\frac{\partial^2 f(S,t)}{\partial S(T)^2}[\sigma^2 S^2\varphi^2\phi+S(t)\varphi\phi(q-1)]+\frac{\partial f(S,t)}{\partial S}(\mu-\lambda k)S(t)\varphi\phi=0 \qquad (24)$$

Using equation (22) and substituting $\phi(S)$ in the first term of equation (24) we have;

$$\frac{\partial}{\partial T}\left[C(t,T)^{-1}\frac{\partial^2 f}{\partial S^2}\right]-\frac{1}{2}\frac{\partial^2 f(S,t)}{\partial S(T)^2}[\sigma^2 S^2\varphi^2\phi+S(t)\varphi\phi(q-1)]+\frac{\partial f(S,t)}{\partial S}(\mu-\lambda k)S(t)\varphi\phi=0 \quad (25)$$

Using chain rule to differentiate the first term of (25) with respect to $T$ and then expand we get;

$$C(t,T)^{-1}\frac{\partial}{\partial T}\left(\frac{\partial^2 f}{\partial S^2}\right)+r(T)C(t,T)^{-1}\frac{\partial^2 f}{\partial S^2}-\frac{1}{2}\frac{\partial^2 f(S,t)}{\partial S(T)^2}[\sigma^2 S^2\varphi^2\phi+S(t)\varphi\phi(q-1)]+\frac{\partial f(S,t)}{\partial S}(\mu-\lambda k)S(t)\varphi\phi=0 \qquad (26)$$

Substituting for $\phi$, multiplying by $C(t,T)$ then integrating once with respect to $S$ we have;

$$\frac{\partial}{\partial T}\left(\frac{\partial f}{\partial S}\right)+r(T)\frac{\partial f}{\partial S}-\frac{1}{2}\frac{\partial f(S,t)}{\partial S}[\sigma^2 S^2\varphi^2+S(t)\varphi\phi(q-1)]\frac{\partial^2 f}{\partial S^2}+(\mu-\lambda k)S(t)\varphi\frac{\partial f}{\partial S}=\alpha(T),$$

Where $\alpha(T)$ is the constant of integration. (27)

Integrating again w.r.t $S$ we get;

$$\frac{\partial f}{\partial T}+r(T)f-\frac{1}{2}[\sigma^2 S^2\varphi^2+S(t)\varphi(q-1)]\frac{\partial^2 f}{\partial S^2}+(\mu-\lambda k)S(t)\varphi\frac{\partial f}{\partial S}=\alpha(T)S+\beta(T), \qquad (28)$$

Where $\beta(T)$ is the costant of integration relating to the second integration. As per Dupire's approach, it is assumed that all the terms on the left-hand side of equation (28) decay when $S$ tends to $+\infty$ so that $\alpha(T)=\beta(T)=0$ hence equation (28) becomes

$$\frac{\partial f}{\partial T}+r(T)f+(\mu-\lambda k)S(t)\varphi\frac{\partial f}{\partial S}-\frac{1}{2}[\sigma^2 S^2\varphi^2+S(t)\varphi(q-1)]\frac{\partial^2 f}{\partial S^2}=0 \qquad (29)$$

Finally solving for volatility model $\sigma$, we have;

$$\sigma = \sqrt{2\left\{\frac{\frac{\partial f}{\partial T}+r(T)f+(\mu-\lambda k)S(t)\varphi\frac{\partial f}{\partial S}-\frac{1}{2}[S(t)\varphi(q-1)]\frac{\partial^2 f}{\partial S^2}}{S^2\varphi^2\frac{\partial^2 f}{\partial S^2}}\right\}} \quad (30)$$

## CONCLUSION

In this paper we have developed a volatility model using European logistic-type Brownian motion when asset price has discontinuity. It is hoped that the results obtained are useful to long term investors before making decisions on profitability of trading strategies.

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