

# An Alternative Method of Constructing Skew-Hadamard Matrices of Order $2^n$

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**Abstract-** In this paper we proposed an alternative method of constructing skew-Hadamard matrices of order  $2^n$ . The doubling construction method can be used to construct an infinite number of skew-Hadamard matrices of order  $2^n (n \geq 2)$ .

**Indexed Terms-** Hadamard matrix, skew-Hadamard matrix

## I. INTRODUCTION

A matrix  $H$  of order  $n$  with entries  $\pm 1$  and satisfying  $HH^T = nI_n$ , where  $H^T$  is the transpose of  $H$  and  $I_n$  is the identity matrix of order  $n$  is called a Hadamard matrix of order  $n$ . It is well known that the order of a Hadamard matrix is 1, 2 or a multiple of 4 [1]. Hadamard matrices have been used in many applications in different fields, such as Olivia MFSK, Balanced repeated replication, Hadamard transform, etc.

Hadamard matrices can be constructed in many ways. A new Hadamard matrix can be obtained from a known Hadamard matrix using the method known as the Sylvester construction. If  $H_n$  is an  $n \times n$  Hadamard matrix, then a  $2n \times 2n$  matrix  $H_{2n}$  can be defined by  $H_{2n} = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}$ . Among these, Skew-Hadamard matrices are of special interest. If  $H$  is a Hadamard matrix and  $H + H^T = 2I_n$ , then  $H$  is called a Skew-Hadamard matrix. Any square matrix  $H$  of order  $n$  is symmetric if  $H = H^T$  and skew symmetric if  $H + H^T = 0$ . Skew-Hadamard matrices are used to construct several combinatorial objects, such as association schemes, self-dual codes, strongly regular graphs etc. [2]. There are various methods to construct skew-Hadamard matrices. But there are many of orders of skew-Hadamard matrices those have not been constructed yet.

Though the constructions of most Hadamard matrices are known, not all of those give skew-Hadamard matrices [3].

Goethals and Seidel proposed a new method to construct skew-Hadamard matrices using four square circulant matrices [4]. Wallis constructed skew-Hadamard matrices of order 92 and he proposed the doubling construction method to construct these matrices [5]. In 1972, Wallis and Whiteman proposed a method to construct unknown skew-Hadamard matrices using four incidence matrices [6].

In this paper we propose an alternative method to construct skew-Hadamard matrices of order  $2^n$ .

## II. MATERIAL AND METHODS

Consider a  $2 \times 2$  matrix  $H_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ . Then  $H_2$  satisfies  $H_2 H_2^T = 2I_2$  and  $H_2 + H_2^T = 2I_2$ . Therefore,  $H_2$  is both Hadamard and skew-Hadamard.

Let's define  $H_4$  as  $H_4 = \begin{bmatrix} H_2 & H_2 \\ -H_2^T & H_2^T \end{bmatrix}$ . Then,  $H_4$  is both Hadamard and skew-Hadamard.

This doubling construction can be used to construct skew-Hadamard matrices of order  $2^n$ .

## III. RESULTS AND DISCUSSION

Consider the  $2 \times 2$  matrix  $H_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ .

Let  $H_4 = \begin{bmatrix} H_2 & H_2 \\ -H_2^T & H_2^T \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$  and

$$H_4^T = \begin{bmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \end{bmatrix}$$

(Here, we denote -1 by - sign.)

Then,  $H_4H_4^T = 4I_4$  and  $H_4 + H_4^T = 2I_4$ .  
Therefore,  $H_4$  is both Hadamard and skew-Hadamard.

Let

$$H_8 = \begin{bmatrix} H_4 & H_4 \\ -H_4^T & H_4^T \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 \\ -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

and

$$H_8^T = \begin{bmatrix} 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

Then,  $H_8H_8^T = 4I_8$  and  $H_8 + H_8^T = 2I_8$ .  
Therefore,  $H_8$  is both Hadamard and skew-Hadamard.  
Generalization of this construction is given by the following Theorem.

Theorem:

Let  $H_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$  be a skew Hadamard matrix of order 2. Then,

$$H_{2^n} = \begin{bmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ -H_{2^{n-1}}^T & H_{2^{n-1}}^T \end{bmatrix}$$

is a skew-

Hadamard matrix of order  $2^n$ .

Proof

Since  $H_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$  is a skew Hadamard matrix of order 2,  $H_2$  satisfies  $H_2H_2^T = 2I_2$  and  $H_2 + H_2^T = 2I_2$ .

Consider  $H_4 = \begin{bmatrix} H_2 & H_2 \\ -H_2^T & H_2^T \end{bmatrix}$ . Then,

$$H_4^T = \begin{bmatrix} H_2^T & -H_2 \\ H_2^T & H_2 \end{bmatrix}$$

Note that

$$H_4H_4^T = \begin{bmatrix} H_2 & H_2 \\ -H_2^T & H_2^T \end{bmatrix} \begin{bmatrix} H_2^T & -H_2 \\ H_2^T & H_2 \end{bmatrix}$$

$$= \begin{bmatrix} H_2H_2^T + H_2H_2^T & -H_2H_2 + H_2H_2 \\ -H_2H_2^T + H_2H_2^T & H_2H_2^T + H_2H_2^T \end{bmatrix}$$

$$= \begin{bmatrix} 2I_2 + 2I_2 & 0 \\ 0 & 2I_2 + 2I_2 \end{bmatrix} = \begin{bmatrix} 4I_2 & 0 \\ 0 & 4I_2 \end{bmatrix} = 4I_4$$

and

$$H_4 + H_4^T = \begin{bmatrix} H_2 & H_2 \\ -H_2^T & H_2^T \end{bmatrix} + \begin{bmatrix} H_2^T & -H_2 \\ H_2^T & H_2 \end{bmatrix}$$

$$= \begin{bmatrix} H_2 + H_2^T & H_2 - H_2 \\ -H_2^T + H_2^T & H_2^T + H_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2I_2 & 0 \\ 0 & 2I_2 \end{bmatrix} = 2 \begin{bmatrix} I_2 & 0 \\ 0 & I_2 \end{bmatrix} = 2I_4.$$

i.e.,  $H_4H_4^T = 4I_4$  and  $H_4 + H_4^T = 2I_4$ . Here "0" denotes the zero matrix of the same order.

Therefore,  $H_4$  is Hadamard and skew-Hadamard.

Now consider,  $H_{2^n} = \begin{bmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ -H_{2^{n-1}}^T & H_{2^{n-1}}^T \end{bmatrix}$ .

Then,  $H_{2^n}^T = \begin{bmatrix} H_{2^{n-1}}^T & -H_{2^{n-1}} \\ H_{2^{n-1}}^T & H_{2^{n-1}} \end{bmatrix}$ .

Note that

$$H_{2^n}H_{2^n}^T = \begin{bmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ -H_{2^{n-1}}^T & H_{2^{n-1}}^T \end{bmatrix} \begin{bmatrix} H_{2^{n-1}}^T & -H_{2^{n-1}} \\ H_{2^{n-1}}^T & H_{2^{n-1}} \end{bmatrix}$$

$$= \begin{bmatrix} H_{2^{n-1}}H_{2^{n-1}}^T + H_{2^{n-1}}H_{2^{n-1}}^T & -H_{2^{n-1}}H_{2^{n-1}} + H_{2^{n-1}}H_{2^{n-1}} \\ -H_{2^{n-1}}H_{2^{n-1}}^T + H_{2^{n-1}}H_{2^{n-1}}^T & H_{2^{n-1}}H_{2^{n-1}}^T + H_{2^{n-1}}H_{2^{n-1}}^T \end{bmatrix}$$

$$= \begin{bmatrix} 2^n I_{2^{n-1}} & 0 \\ 0 & 2^n I_{2^{n-1}} \end{bmatrix} = 2^n \begin{bmatrix} I_{2^{n-1}} & 0 \\ 0 & I_{2^{n-1}} \end{bmatrix}$$

and

$$H_{2^n} + H_{2^n}^T = \begin{bmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ -H_{2^{n-1}}^T & H_{2^{n-1}}^T \end{bmatrix} + \begin{bmatrix} H_{2^{n-1}}^T & -H_{2^{n-1}} \\ H_{2^{n-1}}^T & H_{2^{n-1}} \end{bmatrix}$$

$$= \begin{bmatrix} H_{2^{n-1}} + H_{2^{n-1}}^T & H_{2^{n-1}} - H_{2^{n-1}} \\ -H_{2^{n-1}}^T + H_{2^{n-1}}^T & H_{2^{n-1}}^T + H_{2^{n-1}} \end{bmatrix}$$

$$= \begin{bmatrix} 2I_{2^{n-1}} & 0 \\ 0 & 2I_{2^{n-1}} \end{bmatrix} = 2 \begin{bmatrix} I_{2^{n-1}} & 0 \\ 0 & I_{2^{n-1}} \end{bmatrix}.$$

i.e.,  $H_{2^n}H_{2^n}^T = 2^n I_{2^n}$  and  $H_{2^n} + H_{2^n}^T = 2I_{2^n}$

Therefore,  $H_{2^n}$  is both Hadamard and skew-Hadamard.

Note:

Consider a  $2 \times 2$  skew-Hadamard matrix  $A_2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = H_2^T$ .

Then  $A_{2^n} = \begin{bmatrix} A_{2^{n-1}}^T & A_{2^{n-1}}^T \\ -A_{2^{n-1}} & A_{2^{n-1}} \end{bmatrix}$  is both Hadamard and skew-Hadamard of order  $2^n$ .

There are various methods to construct skew-Hadamard matrices. In this research, an alternative method was proposed. The doubling construction can be used to construct an infinite number of skew-Hadamard matrices of order  $2^n (n \geq 2)$ .

## CONCLUSION

A special emphasis can be given to Skew-Hadamard matrices among other Hadamard matrices. First,  $2 \times 2$  skew-Hadamard matrices were considered. Using the proposed doubling construction method, our main result shows that skew-Hadamard matrices of orders  $4, 8, 16, 32, 64, \dots, 2^n$  ( $n \geq 2$ ) can be constructed. Further, this construction was illustrated with some examples. As a future work, we are planning to implement a computer programme to construct large skew-Hadamard matrices of order  $2^n$  and unknown skew-Hadamard matrices.

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