Survey on Factors of Complete Graphs and Cube Graphs

G.W.M.M.K. DHEERASINGHE[1], A.P. BATUWITA[2], A.A.I. PERERA[3]

[1] *Department of Oceanography and Marine Geology, Faculty of Fisheries and Marine Sciences & Technology, University of Ruhuna, Sri Lanka.*

[2] *Department of Science and Technology, University of UvaWellassa, Sri Lanka.*

[3] *Department of Mathematics, Faculty of Science, University of Peradeniya, Sri Lanka.*

Abstract- Graph theory is an important area in mathematics with many applications. Also, the graph factorization is one of the most flourishing area in graph theory. A factorization of a graph G is a set of spanning sub-graph of G that are pairwise edge-disjoint and whose union is G. Factorization is one of the most active research areas in graph theory. In this work 1-factors, 2-factors and 3-factors of complete graphs were used to create higher order factors. There are different methods to create factors of a graph. In this work degree factors were used. Since it is difficult to create complete graphs manually when the number of nodes increases, a MATLAB code was created to construct complete graphs and identify their factors.

Indexed Terms- Complete graph, Cube graph, Factors of a graph, Factorization

I. INTRODUCTION

Graph theory has many applications in all disciplines. Factors are one of the most interesting areas in Graph Theory. The concept of factorization of a graph was introduced in 1847 by Kirkman. Factors can be constructed using different methods and the degree factors are used in our work. Further, generalized Hadamard matrices can be used to construct 2-factors of a graph.[1].

In general, a graph is represented as a set of vertices (nodes or points) connected by edges (arcs or lines). Therefore, graphs are structures that used to model pair-wise relations between objects. We provide brief summary of definitions which are necessary for the present work.

Definition 1 (Graph) [2]
A simple graph G consists of a non-empty finite set $\mathrm{V(G)}$ of elements called vertices (or nodes), and a finite set $\mathrm{E(G)}$ of distinct unordered pairs of distinct elements of $\mathrm{V(G)}$ called edges.

Definition 2 (Complete Graph) [2]
A simple graph in which every pair of distinct vertices are connected by an edge is called a complete graph. We denote the complete graph on n vertices by $\mathrm{K_n}$.

Definition 3 (Cube Graph) [2]
A cube graph $\mathrm{Q_k}$ is a graph obtained by labeling all vertices using bit strings of length k and two vertices are adjacent whenever corresponding bit strings differ only at one place. Cube graphs $\mathrm{Q_k}$ has 2^{k} vertices and $\mathrm{k}.2^{\mathrm{k}-1}$ edges. Further, cube graph $\mathrm{Q_k}$ is a regular graph of degree k.

Definition 4 (Degree of a vertex) [3]
The degree of a vertex is the number of edges incident with that vertex.

Definition 5 (n-Factor of a Graph) [4]
A factor that is n-regular is called a n-factor.

Definition 6 (Factorization of a Graph) [4]
If a graph G can be represented as the edge-disjoint union of factors $\mathrm{F_1, F_2, \dots, F_k}$, then $\{\mathrm{F_1, F_2, F_3, \dots, F_k}\}$ is referred as a factorization of a graph G.

II. MATERIAL AND METHODS

If a graph G has a n-factorization, then G satisfies two conditions. First condition is that G must be a collection of edge-disjoint n-factors and secondly, the union of n-factors must be the graph G. Factor of a graph G is a spanning sub graph of G having the same vertex set in G and not necessarily having the all edges as in G. If the degree of a spanning sub-graph of G is 1 in every vertex, then it is called a 1-factor of the graph G. The following example represents the cubic graph with three spanning sub graphs G_1, G_2 and G_3 of degree 1. Therefore G_1, G_2 and G_3 are 1-factors of graph G. Since, the union of these three 1-factors is the graph G, it is called a 1-factorization of G.

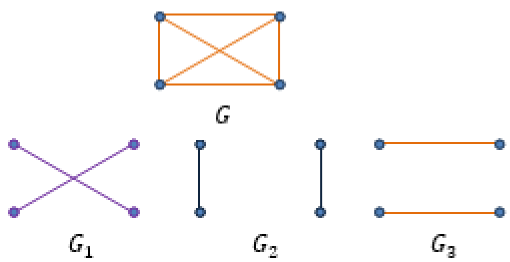

Figure 1: 1-factorization of K_4

Theorem
Cube graph Q_k has k number of 1-factors.

Proof:
Consider the cube graph Q_k with 2^k vertices and $k.2^{k-1}$ edges which is regular of degree k. Since 1-factors are collection of edges of degree 1 and one 1-factor of G has 2^{k-1} edges. The number of 1-factors of G is k. That is $k.2^{k-1}/\ 2^{k-1} = k$.

Example1:

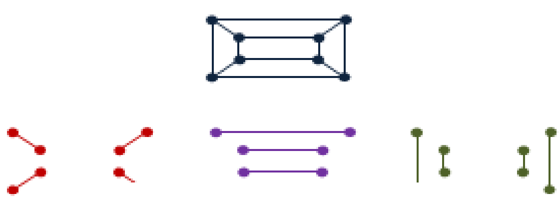

Figure 2: 1-factors of cube graph

2-factor is a sub-graph of a graph G whose each vertex is of degree two and union of these sub-graphs forms the original graph. Vertices of a 2-factorable graph G must have even degree. 3-factorization is the collection of 3-factors. 3-factor is a subgraph of a graph G whose each of the vertices are having degree three. Following figure will show an example of 3-factorization of K_{10}.

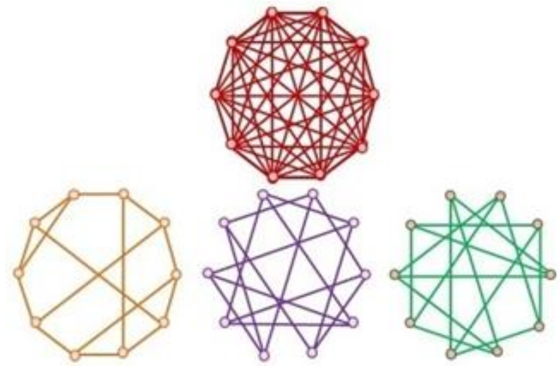

Figure 3: 3-factors of K_{10}

Using 1-factors of 1-factorization we can obtain 2-factors. Following figure will show the construction of 2 factors using 1-factors of K_4.

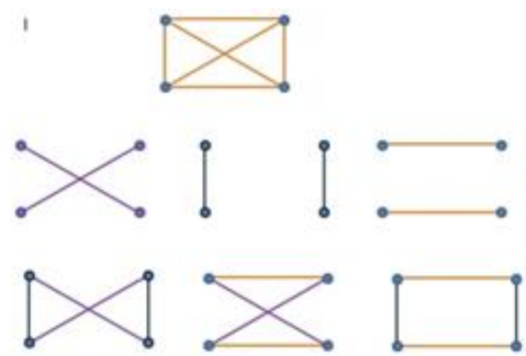

Figure 4: Construction of 2-factors using 1-factors

There are three ways to obtain 2-factors using 1-factors of a 1-factorization of complete graph K_4. The number of 1-factors of K_4 is ${}^3C_2 = 3$.

Consider 1-factorization of K_6. This has five 1-factors. Figure 5 gives 1-factors of K_6 and Figure 6 gives 2-factors constructed using 1-factors of K_6.

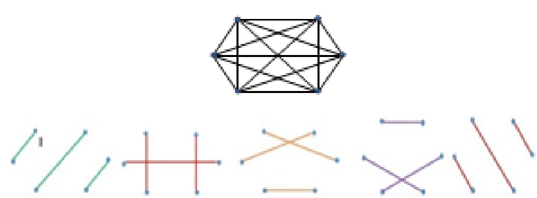

Figure 5: 1-factorization of K_6

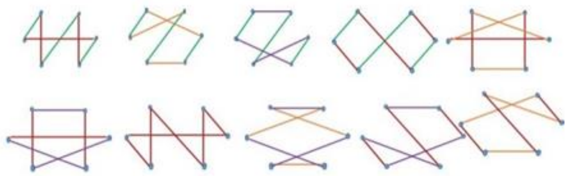

Figure 6: Construction of 2-factors using 1-factors of K_6

There are ${}^5C_2 = 10$ ways to obtain 2-factors using 1-factors of K_6..Here we can have 2-factors only. Further, 2-factorization is not possible since the vertices are having odd degree.

Considering 2-factors of any complete graph in the form K_{2n+1}, $n > 1$, one can construct ${}^mC_2$ number of 4-factors. where m is the number of 2-factors of K_{2n+1},

Example 2: Consider the complete graph K_5. This has 2-factors and ${}^2C_2$ number of 4-factors. Since ${}^2C_2$=1 , K_5 has only one 4-factor and it is the given graph.

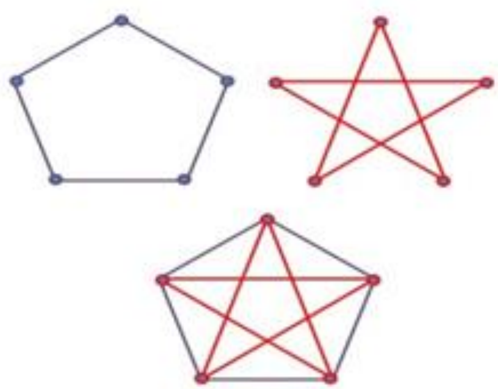

Figure 7: Construction of 4 factors using 2-factors of K_5

Example 3: Consider the complete graph of K_{10}. This has three 3-factors and it is a 3 factorization

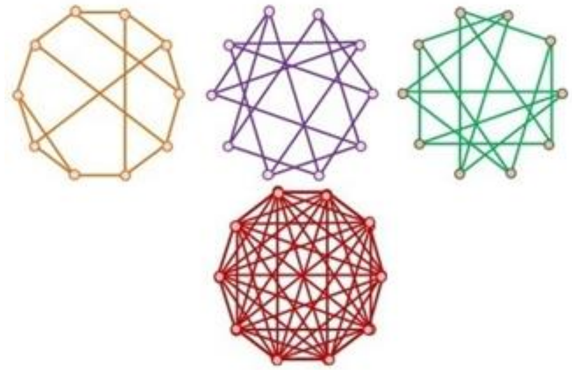

Figure 8: 3-factors of K_{10}

RESULTS AND DISCUSSION

It has been shown that the cube graph Q_k can have k number of 1-factors. Also, it has been clearly described that higher order factors can be constructed using smaller factors of a complete graph. Moreover, from the complete graphs of the form K_{2n} with $n > 1$, using 1-factorization we can have 2-factors but not 2-factorization. To obtain 2-factorization by the using same method, complete graphs of the form K_{2n+1} should be used. Further, 4-factors can be constructed using 2-factorization of any complete graph of the form K_{4n+1} with $n > 1$. For the complete graphs K_{3n+1} with $n > 0$, 3-factors can be constructed and those gives 3-factorization.

Using smaller factors, one can construct higher order factors but not higher order factorizations.

Since it is difficult to construct higher degree factors of complete graphs manually, we created a MATLAB program to draw any complete graph and identify its factors.

CONCLUSION

Factorization is the most fascinating area of graph theory with various applications. It has been shown that cube graph Q_k can have k number of 1-factors and that theorem has been proved. Moreover, 1-factors, 2-factors and 3-factors of complete graphs have been constructed. As future work, we are planning to apply the properties of factors to solve graph theoretic problems.

REFERENCES

[1] W.V..Nishadi and A. A. I. Perera, Generalized Hadamard matrices and 2-factorizaton of Complete Graphs, Asian research Journal of Mathematics, (2020).

[2] J. R. Wilson, *Introduction to Graph Theory*, Longman Group Ltd.4th edition, (1996).

[3] L. G. Jonathan, J. Yellan, *Handbook of Graph Theory*, CRC Press, (2003)

[4] J. Akiyama, M. Kano, *Factors and Factorization of Graphs*, Journal of Graph Theory, Vol 9, pp1-42, (1985)