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On *D-Operator

WANJALA VICTOR[1], A.M. NYONGESA[2]
[1, 2] Department of Mathematics Kibabii University, Kenya.

***Abstract-** In this paper, we introduce the class of *D-Operator a bounded linear operator T is said to be a *D-Operator if $T^{*2}(T^D)^2 = (T^D T^*)^2$. we investigate the basic properties of this class and also show that this class is closed under strong operator topology.*

***Indexed Terms-** D-Operator, *D-Operator, Class (Q), Almost Class (Q), (α, β)-Class (Q), Normal operators, n-Normal, n-D-Operator operators.*

I. INTRODUCTION

Throughout this paper, H is a separable complex Hilbert space, B(H) is the Banach algebra of all bounded linear operators. n-normal if $T^* T^n = T^n T^*$, $T \in B(H)$ is normal if $T^*T = TT^*$, quasinormal if $T(T^*T) = (T^*T)T$. D-Operator if $T^{*2}(T^D)^2 = (T^*T^D)^2$ (1), class (Q) if $T^{*2}T^2 = (T^*T)^2$ (5), n-power class (Q) if $T^{*2}(T^n)^2 = (T^*T^n)^2$ (6), n-D-Operator if $T^{*2}(T^D)^{2n} = (T^*(T^D)^n)^2$, for any positive integer n. We note that n-D-Operator is D-Operator when n=1.

II. MAIN RESULTS

- Definition 1. Let $T \in B(H)$ be Drazin invertible. Then an operator T is called *D-Operator, denoted by, $[^*D]$, if $T^{*2}(T^D)^{2n} = (T^*(T^D)^n)^2$, for any positive integer n.
- Proposition 2. Let $T \in [^*D]$, then the following holds;

i. $\lambda T \in [^*D]$ for every scalar λ.
ii. $S \in [^*D]$ for every $S \in B(H)$ that is unitarily equivalent to T.
iii. The restriction/M of T to any closed subspace M of H which reduces T is in $[^*D]$.
iv. $(T^D) \in [^*D]$.

- Proof.

(i) The proof is trivial.

(ii) Since S is unitarily equivalent to T, there exists a unitary operator $U \in B(H)$ such that $S=UTU^*$. Hence;

$S^{*2n}(S^D)^{2n} = (UT^*U^*)^2(U(T^D)^nU^*)^2$

$= (UT^*U^*)(UT^*U^*)(U(T D)^nU^*)(U(T^D)^nU^*)$
$=UT^*T^*(T^D)^n(T^D)^nU^*$
$= UT^{*2}(T^D)^{2n}U^*$
$=U(T^*(T^D)^n)^2U^*$
$=UT^*(T^D)^nT^*(T^D)^nU^*$
$= (UT^*U^*)(U(T^D)^nU^*)(UT^*U^*)(U(T^D)^nU^*)$
$= S^*(S^D)^nS^*(S^D)^n$
$= (S^*(S^D)^n)^{2.}$

Thus $S \in [^*D]$.

(iii) $(T/M)^{*2}((T/M)D)^{2n} = (T/M)^*(T/M)^*((T/M)^D)^n((T/M)^D)^n$
$= (T^*/M)(T^*/M)((T^D)^n/M)((T^D)^n/M)$
$= (T^*T^*/M)((T^D)^{nT\,D})^n/M)^2$
$= (T^{*2}/M)((T^D)^{2n}/M)$
$= (T^{*2}(T^D)^{2n})/M$
$= (T^*(T^D)^nT^*(T^D)^n)/M$
$= ((T^*(T^D)^n)/M)((T^*(T^D)^n)/M)$
$= ((T^*/M)((T^D)^n/M)(T^*/M)((T^D)^n/M))$
$= ((T^*/M)((T^D)^{n/}M)^2$
$= ((T/M)^*((T/M)^D)^n)^{2.}$

Hence $T/M \in [^*D]$.

(iv) Suppose $T\in [^*D]$, then;
$T^{*2n}(T^D)^{2n} = (T^*(T^D)^n)^2$, hence
$T^*T^*(T^D)^n(T^D)^n = T^*(T^D)^nT^*(T^D)^n$
taking adjoins on both sides
$= ((T^*)^D)^n((T^*)^D)^nTT = ((T^*)^D)^nT((T^*)^D)^nT.$
Thus $(((T^D)^n)^*)^2T^2 = (((T^D)^n)^{*)}T)^2$.
hence $(T D)n \in [^*D]$.

- Proposition 3. The set of all *D-Operators is a closed subset of B(H) on H. Proof.

Let $\{T_q\}$ be a sequence of $[{}^*D]$ operators with $T_q \longrightarrow T$. We have to show that
$T \in [{}^*D]$. Now $T_q \to T$ implies $T_q^* \to T^*$ and $(T_q^D)^n \to (T^D)^n$. Thus $T_q^*(T_q^D)^n \to T^*(T^D)^n$ gives
$(T_q^*(T_q^D)^n)^2 \to (T^*(T^D)^n)^2$ …………. (0.1)

Similarly,
$T_q^{*2} \to T^{*2}$ and $(T_q^D)^{2n} \to (T^D)^{2n}$, thus
$T_q^{*2}(T_q^D)^{2n} \to T^{*2}(T^D)^{2n}$ ……………. (0.2) 3
hence from (0.1) and (0.2) we have;

$$\| T^{*2}(T^D)^{2n} - (T^*(T^D)^n)^2 \|$$

$$= \| T^{*2}(T^D)^{2n} - T_q^{*2}(T_q^D)^{2n} + T_q^{*2}(T_q^D)^{2n} - (T^*(T^D)^n)^2 \|$$

$$\leq \| T^{*2}(T^D)^{2n} - T_q^{*2}(T_q^D)^{2n} \| + \| T_q^{*2}(T_q^D)^{2n} - (T^*(T^D)^n)^2 \|$$

$$= \| T^{*2}(T^D)^{2n} - T_q^{*2}(T_q^D)^{2n} \| + \| T_q^{*2}((T_q^D)^n)^2 - (T^*(T^D)^n)^2 \| \to 0 \text{ as } q \to \infty$$ and thus

$T^{*2}(T^D)^{2n} = (T^*(T^D)^n)^2$ hence $T \in [{}^*D]$.

- Proposition 4. Let $S, T \in [{}^*D]$. If $[S, T] = [S, T^*] = 0$, then $TS \in [{}^*D]$.

Proof
$[S, T] = [S, T^*] = 0$ implies;

$[S, T] = [S^D, T] = [S^*, T^D] = 0$ with $S, T \in [{}^*D]$ we have; $S^{*2}(S^D)^{2n} = (S^*(S^D)^n)^2$ and
$T^{*2}(T^D)^{2n} = (T^*(T^D)^n)^2$, hence

$(TS)^{*2}((TS)^D)^{2n} = (TS)^*(TS)^*(TS)^D(TS)^D$
$= S^*T^*S^*T^*(T^D)^n(S^D)^n(T^D)^n(S^D)^n$
$= S^*S^*(S^D)^n(S^D)^nT^*T^*(T^D)^n(T^D)^n$
$= S^{*2}T^{*2}(S^D)^{2n}(T^D)^{2n}$
$= S^*S^*T^*T^*(S^D)^n(T^D)^n(S^D)^n(T^D)^n$
$= S^*T^*S^*T^*(S^D)^n(T^D)^n(S^D)^n(T^D)^n$
$= (TS)^*(TS)^*((TS)^D)^n)^2$.
Hence $TS \in [{}^*D]$.

- Proposition 5. Let $S, T \in [{}^*D]$. If $TS = ST = 0$, then $S+T \in [{}^*D]$.

Proof.
$S, T \in [{}^*D]$ implies; $S^{*2}(S^D)^{2n} = (S^*(S^D)^n)^2$ and
$T^{*2}(T^D)^{2n} = (T^*(T^D)^n)^2$.
$TS=ST=0$ implies $T^*S^* = S^*T^*$ which further implies $((S + T)^D)^n = (SD)^n + (T^D)^n$. Thus,
$= (S + T)^{*2}((S + T) D)^{2n} = (S + T)^*(S + T)^*((S + T) D)^n ((S + T) D)^n$
$= (S^* + T^*)(S^* + T^*)(S^D + T^D)^n(S^D + T^D)^n$
$= (S^{*2} + T^{*2})((S^D)^{2n} + (T^D)^{2n})$
$= S^{*2}(S^D)^{2n} + T^{*2}(T^D)^{2n}$
$= (S^*(S^D)^n)^2 + (T^*(T^D)^n)^2$
$= (S^*(S^D)^n + T^*(T^D)^n)(S^*(S^D)^n + T^*(T^D)^n)$
$= (S^* + T^*)((S^D)^n + (T^D)^n)(S^* + T^*)((S^D)^n + (T^D)^n)$
$= ((S + T)^*((S + T)^D)^n)^2$.
Hence $S+T \in [{}^*D]$.

Theorem 6. Let $T_{\alpha 1}, T_{\alpha 2} \ldots\ldots\ldots T_{\alpha q} \in [{}^*D]$, then it follows that;
(i) $T_{\alpha 1} \oplus T_{\alpha 2} \oplus \ldots\ldots \oplus T_{\alpha q} \in [nD]$.
(ii) $T_{\alpha 1} \otimes T_{\alpha 2} \otimes \ldots\ldots \otimes T_{\alpha q} \in [nD]$.

Proof. (i) . $T_{\alpha j} \in [nD]$ for all $\alpha j = 1, 2, \ldots\ldots \alpha q$ implies;
$T_{\alpha j}^{*2}(T_{\alpha j}^D)^{2n} = (T_{\alpha j}^*(T_{\alpha j}^D)^n)^2$ thus
$(T_{\alpha 1} \oplus T_{\alpha 2} \oplus \ldots\ldots \oplus T_{\alpha j})^{*2}((T_{\alpha 1} \oplus T_{\alpha 2} \oplus \ldots\ldots \oplus T_{\alpha j})^D)^{2n}$
$= T_{\alpha 1}^{*2}(T_{\alpha 1}^D)^{2n} \oplus T_{\alpha 2}^{*2}(T_{\alpha 2}^D)^{2n} \oplus \ldots\ldots \oplus T_{\alpha j}^{*2}(T_{\alpha j}^D)^{2n}$
$= (T_{\alpha 1}^*(T_{\alpha 1}^D)^n)^2 \oplus (T_{\alpha 2}^*(T_{\alpha 2}^D)^n)^2 \oplus \ldots\ldots \oplus (T_{\alpha j}^*(T_{\alpha j}^D)^n)^2$
$= T_{\alpha 1}^*(T_{\alpha 1}^D)^n T_{\alpha 1}^*(T_{\alpha 1}^D)^n \oplus T_{\alpha 2}^*(T_{\alpha 2}^D)^n T_{\alpha 2}^*(T_{\alpha 2}^D)^n \oplus \ldots\ldots\ldots \oplus T_{\alpha j}^*(T_{\alpha j}^D)^n T_{\alpha j}^*(T_{\alpha j}^D)^n$
$= T_{\alpha 1}^*(T_{\alpha 1}^D)^n \oplus T_{\alpha 2}^*(T_{\alpha 2}^D)^n \oplus \ldots\ldots\ldots \oplus T_{\alpha j}^*(T_{\alpha j}^D)^n$
$= ((T_{\alpha 1}^* \oplus T_{\alpha 2}^* \oplus \ldots\ldots \oplus T_{\alpha j}^*)((T_{\alpha 1}^D)^n \oplus (T_{\alpha 2}^D)^n \oplus \ldots\ldots \oplus (T_{\alpha j}^D)^n))$
$= ((T_{\alpha 1} \oplus T_{\alpha 2} \oplus \ldots\ldots \oplus T_{\alpha j})^*((T_{\alpha 1} \oplus T_{\alpha 2} \oplus \ldots\ldots \oplus T_{\alpha j})^D)^n)^2$

(v) The proof for (ii) follows similarly.

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