On (N+K, M)- D-Operator Operators

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Abstract- In this paper, we generalize the class of D-Operator by extending this study to n-D-Operator and investigate the basic properties of this class. We also show that this class is closed under strong operator topology.

Indexed Terms- D-Operator, Class (Q), Almost Class (Q), (α, β)-Class (Q), Normal operators, n-Normal, n-D-Operator operators.

I. INTRODUCTION

Throughout this paper, H is a separable complex Hilbert space, B(H) is the Banach algebra of all bounded linear operators. n-normal if $T^* T^n = T^n T^*$, $T \in B(H)$ is normal if $T^*T = TT^*$, quasinormal if $T(T^*T) = (T^*T)T$. D-Operator if $T^{*2}(T^D)^2 = (T^*T^D)^2$ (1), class (Q) if $T^{*2}T^2 = (T^*T)^2$ (5), n-power class (Q) if $T^{*2}(T^n)^2 = (T^*T^n)^2$ (6), n-D-Operator if $T^{*2}(T^D)^{2n} = (T^*(T^D)^n)^2$, for any positive integer n. We note that n-D-Operator is D-Operator when n=1.

II. MAIN RESULTS

- Definition 1. Let $T \in B(H)$ be Drazin invertible. Then an operator T is called (n+k, m)-power- D-Operator, denoted by, [n+k, mD], if $T^{*2m}(T^D)^{2(n+k)} = (T^{*m}(T^D)^{n+k})^2$, for any positive integers n and m.

- Proposition 2. Let $T \in$ [n+k, mD], then the following holds;

i. $\lambda T \in$ [n+k, mD] for every scalar λ.
ii. $S \in$ [n+k, mD] for every $S \in B(H)$ that is unitarily equivalent to T.
iii. The restriction/M of T to any closed subspace M of H which reduces T is in [n+k, mD].
iv. $(T^D)^{n+k} \in$ [n+kD].

Proof.

i. The proof is trivial.

ii. Since S is unitarily equivalent to T, there exists a unitary operator $U \in B(H)$ such that $S = UTU^*$. Hence;

$S^{*2m}(S^D)^{2(n+k)} = (UT^{*m}U^*)^2(U(T^D)^{n+k}U^*)^2$

$= (UT^{*m}U^*)(UT^{*m}U^*)(U(T^D)^{n+k}U^*)(U(T^D)^{n+k}U^*)$
$= UT^*T^{*m}(T^D)^{n+k}(T^D)^{n+k}U^*$
$= UT^{*2m}(T^D)^{2(n+k)}U^*$
$= U(T^{*m}(T^D)^{n+k})^2U^*$
$= UT^{*m}(T^D)^{n+k}T^{*m}(T^D)^{n+k}U^*$
$= (UT^{*m}U^*)(U(T^D)^{n+k}U^*)(UT^{*m}U^*)(U(T^D)^{n+k}U^*)$
$= S^{*m}(S^D)^{n+k}S^{*m}(S^D)^{n+k}$
$= (S^{*m}(S^D)^n)^{2.}$

Thus $S \in$ [n+k, mD].

iii. $(T/M)^{*2m}((T/M)D)^{2(n+k)} = (T/M)^{*m}(T/M)^{*m}((T/M)^D)^{n+k}((T/M)^D)^{n+k}$
$= (T^{*m}/M)(T^{*m}/M)((T^D)^{n+k}/M)((T^D)^{n+k}/M)$
$= (T^{*m}T^{*m}/M)((T^D)^{n+k}T^D)^{n+k}/M)^2$
$= (T^{*2m}/M)((T^D)^{2(n+k)}/M)$
$= (T^{*2m}(T^D)^{2(n+k)})/M$
$= (T^{*m}(T^D)^{n+k}T^{*m}(T^D)^{n+k})/M$
$= ((T^{*m}(T^D)^{n+k})/M)((T^{*m}(T^D)^{n+k})/M)$
$= ((T^{*m}/M)((T^D)^{n+k}/M)(T^{*m}/M)((T^D)^{n+k}/M))$
$= ((T^{*m}/M)((T^D)^{n+k/}M)^2$
$= ((T/M)^{*m}((T/M)^D)^{n+k})^{2.}$

Hence $T/M \in$ [n+k, mD].

iv. Suppose $T \in$ [n+k, mD], then;
$T^{*2n}(T^D)^{2(n+k)} = (T^*(T^D)^{n+k})^2$, hence
$T^{*m}T^{*m}(T^D)^{n+k}(T^D)^{n+k} = T^{*m}(T^D)^{n+k}T^{*m}(T^D)^{n+k}$
taking adjoins on both sides

$= ((T^*)^D)^{n+k}((T^*)^D)^{n+k}T^mT^m = ((T^*)^D)^{n+k}T^m((T^*)^D)^{n+k}T^m$.
Thus $(((T^D)^{n+k})^*)^2T^{2m} = (((T^D)^{n+k})^*)T)^{2m}$. hence $(T^D)^{n+k} \in$[n+k, mD].

- Proposition 3. The set of all (n+k, m)-D-Operators is a closed subset of B(H) on H. Proof.

Let $\{T_q\}$ be a sequence of [n+k, mD] operators with $T_q \longrightarrow T$. We have to show that $T \in [n+k, mD]$. Now $T_q \to T$ implies $T_q^{*m} \to T^{*m}$ and $(T_q^D)^{n+k} \to (T^D)^{n+k}$. Thus $T_q^{*m}(T_q^D)^{n+k} \to T^{*m}(T^D)^{n+k}$ gives

$(T_q^{*m}(T_q^D)^{n+k})^2 \to (T^{*m}(T^D)^{n+k})^2$ ………… (0.1)

Similarly,

$T_q^{*2m} \to T^{*2m}$ and $(T_q^D)^{2(n+k)} \to (T^D)^{2(n+k)}$, thus

$T_q^{*2m}(T_q^D)^{2(n+k)} \to T^{*2m}(T^D)^{2(n+k)}$ ……………. (0.2)

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hence from (0.1) and (0.2) we have;

$\| T^{*2m}(T^D)^{2(n+k)} - (T^{*m}(T^D)^{n+k})^2 \|$

$= \| T^{*2m}(T^D)^{2(n+k)} - T_q^{*2m}(T_q^D)^{2(n+k)} + T_q^{*2m}(T_q^D)^{2(n+k)} - (T^{*m}(T^D)^{n+k})^2 \|$

$\leq \| T^{*2m}(T^D)^{2(n+k)} - T_q^{*2m}(T_q^D)^{2(n+k)} \| + \| T_q^{*2m}(T_q^D)^{2(n+k)} - (T^{*m}(T^D)^{n+k})^2 \|$

$= \| T^{*2m}(T^D)^{2(n+k)} - T_q^{*2m}(T_q^D)^{2(n+k)} \| + \| T_q^{*2m}((T_q^D)^{n+k})^2 - (T^{*m}(T^D)^{n+k})^2 \| \to 0$ as $q \to \infty$ and thus

$T^{*2m}(T^D)^{2(n+k)} = (T^{*m}(T^D)^{n+k})^2$ hence $T \in [n+k, mD]$.

- Proposition 4. Let $S, T \in [n+k, mD]$. If $[S, T] = [S, T^*] = 0$, then $TS \in [n+k, mD]$.

Proof

$[S, T] = [S, T^*] = 0$ implies;

$[S, T] = [S^D, T] = [S^*, T^D] = 0$ with $S, T \in [n+k, mD]$ we have; $S^{*2m}(S^D)^{2(n+k)} = (S^{*m}(S^D)^{n+k})^2$ and

$T^{*2m}(T^D)^{2(n+k)} = (T^{*m}(T^D)^{n+k})^2$, hence

$(TS)^{*2m}((TS)^D)^{2(n+k)} = (TS)^{*m}(TS)^{*m}(TS)^D(TS)^D$

$= S^{*m}T^{*m}S^{*m}T^{*m}(T^D)^{n+k}(S^D)^{n+k}(T^D)^{n+k}(S^D)^{n+k}$

$= S^{*m}S^{*m}(S^D)^{n+k}(S^D)^{n+k}T^{*m}T^{*m}(T^D)^{n+k}(T^D)^{n+k}$

$= S^{*2m}T^{*2m}(S^D)^{2(n+k)}(T^D)^{2(n+k)}$

$= S^{*m}S^{*m}T^{*m}T^{*m}(S^D)^{n+k}(T^D)^{n+k}(S^D)^{n+k}(T^D)^{n+k}$

$= S^{*m}T^{*m}S^{*m}T^{*m}(S^D)^{n+k}(T^D)^{n+k}(S^D)^{n+k}(T^D)^{n+k}$

$= (TS)^{*m}(TS)^{*m}((TS)^D)^{n+k})^2$.

Hence $TS \in [n+k, mD]$.

Proposition 5. Let $S, T \in [n+k, mD]$. If $TS = ST = 0$, then $S+T \in [n+k, mD]$.

Proof.

$S, T \in [n+k, mD]$ implies; $S^{*2m}(S^D)^{2(n+k)} = (S^{*m}(S^D)^{n+k})^2$ and

$T^{*2m}(T^D)^{2(n+k)} = (T^{*m}(T^D)^{n+k})^2$.

$TS=ST=0$ implies $T^*S^* = S^*T^*$ which further implies $((S + T)^D)^{n+k} = (SD)^{n+k} + (T^D)^{n.+k}$ Thus,

$= (S + T)^{*2}((S + T)D)^{2(n+k)} = (S + T)^*(S + T)^*((S + T)D)^{n+k}((S + T)D)^{n+k}$

$= (S^{*m} + T^{*m})(S^{*m} + T^{*m})(S^D + T^D)^{n+k}(S^D + T^D)^{n+k}$

$= (S^{*2m} + T^{*2m})((S^D)^{2(n+k)} + (T^D)^{2(n+k)})$

$= S^{*2m}(S^D)^{2(n+k)} + T^{*2m}(T^D)^{2(n+k)}$

$= (S^{*m}(S^D)^{n+k})^2 + (T^{*m}(T^D)^{n+k})^2$

$= (S^{*m}(S^D)^{n+k} + T^{*m}(T^D)^{n+k})(S^{*m}(S^D)^{n+k} + T^{*m}(T^D)^{n+k})$

$= (S^{*m} + T^{*m})((S^D)^{n+k} + (T^D)^{n+k})(S^{*m} + T^{*m})((S^D)^{n+k} + (T^D)^{n+k})$

$= ((S + T)^{*m}((S + T)^D)^{n+k})^2$.

Hence $S+T \in [n+k, mD]$.

- Theorem 6. Let $T_{\alpha 1}, T_{\alpha 2} \ldots\ldots\ldots T_{\alpha q} \in [n+k, mD]$, then it follows that;

i. $T_{\alpha 1} \oplus T_{\alpha 2} \oplus \ldots\ldots \oplus T_{\alpha q} \in [n+k, mD]$.

ii. $T_{\alpha 1} \otimes T_{\alpha 2} \otimes \ldots\ldots \otimes T_{\alpha q} \in [n+k, mD]$.

Proof.

$T_{\alpha j} \in [n+k, mD]$ for all $\alpha j = 1, 2, \ldots\ldots \alpha q$ implies;

$T_{\alpha j}^{*2m}(T_{\alpha j}^D)^{2(n+k)} = (T_{\alpha j}{*m}(T_{\alpha j}^D)^{n+k})^2$

thus

$(T_{\alpha 1} \oplus T_{\alpha 2} \oplus \ldots\ldots \oplus T_{\alpha j})^{*2m}((T_{\alpha 1} \oplus T_{\alpha 2} \oplus \ldots\ldots \oplus T_{\alpha j})^D)^{2(n+k)}$

$= T_{\alpha 1}^{*2m}(T_{\alpha 1}^D)^{2(n+k)} \oplus T_{\alpha 2}^{*2m}(T_{\alpha 2}^D)^{2(n+k)} \oplus \ldots\ldots \oplus T_{\alpha j}^{*2m}(T_{\alpha j}^D)^{2(n+k)}$

$= (T_{\alpha 1}^{*m}(T_{\alpha 1}^D)^{n+k})^2 \oplus (T_{\alpha 2}^{*m}(T_{\alpha 2}^D)^{n+k})^2 \oplus \ldots\ldots \oplus (T_{\alpha j}^{*m}(T_{\alpha j}^D)^{n+k})^2$

$= T_{\alpha 1}^{*m}(T_{\alpha 1}^D)^{n+k}T_{\alpha 1}^{*m}(T_{\alpha 1}^D)^{n+k} \oplus T_{\alpha 2}^{*m}(T_{\alpha 2}^D)^{n+k}T_{\alpha 2}^{*m}(T_{\alpha 2}^D)^{n+k} \oplus \ldots\ldots\ldots \oplus T_{\alpha j}^{*m}(T_{\alpha j}^D)^{n+k}T_{\alpha j}^{*m}(T_{\alpha j}^D)^{n+k}$

$= T_{\alpha 1}^{*m}(T_{\alpha 1}^D)^{n+k} \oplus T_{\alpha 2}^{*m}(T_{\alpha 2}^D)^{n+k} \oplus \ldots\ldots\ldots \oplus T_{\alpha j}^{*m}(T_{\alpha j}^D)^{n+k}$

$= ((T_{\alpha 1}{}^{*m} \oplus T_{\alpha 2}{}^{*m} \oplus \ldots\ldots \oplus T_{\alpha j}{}^{*m}) ((T_{\alpha 1}{}^{D})^{n+k} \oplus (T_{\alpha 2}{}^{D})^{n+k} \oplus \ldots\ldots \oplus (T_{\alpha j}{}^{D})^{n+k}))$

$= ((T_{\alpha 1} \oplus T_{\alpha 2} \oplus \ldots\ldots \oplus T_{\alpha j})^{*m} ((T_{\alpha 1} \oplus T_{\alpha 2} \oplus \ldots\ldots \oplus T_{\alpha j})^{D})^{n+k})^{2}$

iii. The proof for (ii) follows similarly.

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