

On (N+K)- Power- D-Operator

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Abstract- In this paper, we introduce the class of (N+K)-power -D-operator acting on the Hilbertspace H over the complex plane. A bounded linear operator T is said to be an (N+K)-power -D-operator if $T^{*2}(T^D)^{2(n+k)} = (T * (T^D)^{n+k})^2$ for positive integers n and k and where T^D is the Drazin inverse of T. We investigate the basic behavior of this class of operator.

Indexed Terms- Normal operators, D-Operator, Almost Class (Q), quasi -class (Q) operators, N quasi D-operator.

I. INTRODUCTION

H denotes the separable complex Hilbert space in this paper, while B(H) is the usual Banach algebra of all bounded linear operators on H. Let $T \in B(H)$, Drazin inverse of T is an operator $T^D \in B(H)$, such that $TT^D = T^DT$, $T^D = T^D TT^D$ and $T^{k+1}T^D = T^k$ provided it exists. An operator $T \in B(H)$ is said to be D-Operator if $T^{*2}(T^D)^2 = (T * T^D)^2$, this was covered by Abood and Kadhim in (1), N quasi D- Operator was covered by Wanjala Victor and A.M. Nyongesa in (6), a bounded linear operator T is said to be N Quasi D-operator if $T^{*2}(T^D)^2 = N(T * T^D)^2$, M Quasi class (Q) if $T^{*2}(T^D)^2 = M(T * T^D)^2$ (5), class (Q) if $T^{*2}(T^D)^2 = (T * T^D)^2$ (4), Quasiclass (Q) if $T^{*2}(T^D)^2 = (T * T^D)^2$, for a bounded linear operator N. Let $T = \xi + i\zeta$, with $\xi = \text{Re}(T) = \frac{T + T^*}{2}$ and $\zeta = \text{Im}(T) = \frac{T - T^*}{2i}$. We shall simply denote $U^2 = (T * T^D)^2$ and $V^2 = T^{*2}(T^D)^2$ where C and V are non-negative definite.

II. MAIN RESULTS

Definition 1. Let $T \in B(H)$ be Drazin invertible, an operator T is called (N+K)-power -D-operator if $T^{*2}(T^D)^{2(n+k)} = (T * (T^D)^{n+k})^2$ for positive integers n and k.

Proposition 2. Let T be (n+k)-power D-operator, then the following holds;

- i. $\lambda T \in (n+k)$ -D-operator for every scalar λ .

- ii. $S \in (n+k)$ -D-operator for every $S \in B(H)$ that is unitarily equivalent to T.
- iii. The restriction/M of T to any closed subspace M of H which reduces T is in (n+k)-D-operator.
- iv. $(T^D)^{n+k} \in (n+k)$ -D-operator.

Proof.

- i. The proof is trivial.
- ii. Since S is unitarily equivalent to T, there exists a unitary operator $U \in B(H)$ such that $S = UTU^*$. Hence;

$$\begin{aligned} S^{*2n}(S^D)^{2(n+k)} &= (UT^*U^*)^2(U(T^D)^{n+k}U^*)^2 \\ &= (UT^*U^*)(UT^*U^*)(U(T^D)^{n+k}U^*)(U(T^D)^{n+k}U^*) \\ &= UT^*T^*(T^D)^{n+k}(T^D)^{n+k}U^* \\ &= UT^*(T^D)^{2(n+k)}U^* \\ &= U(T^*(T^D)^{n+k})^2U^* \\ &= UT^*(T^D)^{n+k}T^*(T^D)^{n+k}U^* \\ &= (UT^*U^*)(U(T^D)^{n+k}U^*)(UT^*U^*)(U(T^D)^{n+k}U^*) \\ &= S^*(S^D)^{n+k}S^*(S^D)^{n+k} \\ &= (S^*(S^D)^{n+k})^2 \end{aligned}$$

Thus $S \in (n+k)$ -D-operator.

$$\begin{aligned} \text{iii. } (T/M)^{*2}((T/M)^D)^{2(n+k)} &= (T/M)^*(T/M)^*((T/M)^D)^{n+k}((T/M)^D)^{n+k} \\ &= (T^*/M)(T^*/M)((T^D)^{n+k}/M)((T^D)^{n+k}/M) \\ &= (T^*T^*/M)((T^D)^{n+k}(T^D)^{n+k}/M)^2 \\ &= (T^{*2}/M)((T^D)^{2(n+k)}/M) \\ &= (T^{*2}(T^D)^{2(n+k)})/M \\ &= (T^*(T^D)^{n+k}T^*(T^D)^{n+k})/M \\ &= ((T^*(T^D)^{n+k})/M)((T^*(T^D)^{n+k})/M) \\ &= ((T^*/M)((T^D)^{n+k}/M)(T^*/M)((T^D)^{n+k}/M)) \\ &= ((T^*/M)((T^D)^{n+k}/M))^2 \\ &= ((T/M)^*((T/M)^D)^{n+k})^2 \end{aligned}$$

Hence $T/M \in (n+k)$ -D-operator.

- iv. Suppose $T \in (n+k)$ -D-operator, then; $T^{*2n}(T^D)^{2(n+k)} = (T^*(T^D)^{n+k})^2$, hence $T^*T^*(T^D)^{n+k}(T^D)^{n+k} = T^*(T^D)^{n+k}T^*(T^D)^{n+k}$ taking adjoints on both sides

$$= ((T^*)^D)^{n+k} ((T^*)^D)^{n+k} T T = ((T^*)^D)^{n+k} T ((T^*)^D)^{n+k} T.$$

Thus $((T^D)^{n+k})^* T^2 = (((T^D)^{n+k})^* T)^2$.

hence $(T^D)^{n+k} \in (n+k)$ -D-operator.

Proposition 3. The set of all $(n+k)$ -D-operator is a closed subset of $B(H)$ on H .

Proof.

Let $\{T_q\}$ be a sequence of $(n+k)$ -D-operator operators with $T_q \rightarrow T$. We have to show that

$T \in (n+k)$ -D-operator. Now $T_q \rightarrow T$ implies $T_q^* \rightarrow T^*$ and $(T_q^D)^{n+k} \rightarrow (T^D)^{n+k}$. Thus $T_q^*(T_q^D)^{n+k} \rightarrow T^*(T^D)^{n+k}$ gives

$$(T_q^*(T_q^D)^{n+k})^2 \rightarrow (T^*(T^D)^{n+k})^2 \dots\dots\dots (0.1)$$

Similarly,

$$T_q^2 \rightarrow T^2 \text{ and } (T_q^D)^{2(n+k)} \rightarrow (T^D)^{2(n+k)}, \text{ thus}$$

$$T_q^2 (T_q^D)^{2(n+k)} \rightarrow T^2 (T^D)^{2(n+k)} \dots\dots\dots (0.2)$$

hence from (0.1) and (0.2) we have;

$$\begin{aligned} & \| T^2 (T^D)^{2(n+k)} - (T^*(T^D)^{n+k})^2 \| \\ &= \| T^2 (T^D)^{2(n+k)} - T_q^2 (T_q^D)^{2(n+k)} + T_q^2 (T_q^D)^{2(n+k)} - (T^*(T^D)^{n+k})^2 \| \\ &\leq \| T^2 (T^D)^{2(n+k)} - T_q^2 (T_q^D)^{2(n+k)} \| + \| T_q^2 (T_q^D)^{2(n+k)} - (T^*(T^D)^{n+k})^2 \| \\ &= \| T^2 (T^D)^{2(n+k)} - T_q^2 (T_q^D)^{2(n+k)} \| + \| T_q^2 ((T_q^D)^{n+k})^2 - (T^*(T^D)^{n+k})^2 \| \rightarrow 0 \text{ as } q \rightarrow \infty \text{ and thus} \end{aligned}$$

$T^2 (T^D)^{2(n+k)} = (T^*(T^D)^{n+k})^2$ hence $T \in (n+k)$ -D-operator.

• Proposition 4. Let $S, T \in (n+k)$ -D-operator. If $[S, T] = [S, T^*] = 0$, then $TS \in (n+k)$ -D-operator.

Proof

$$[S, T] = [S, T^*] = 0 \text{ implies;}$$

$[S, T] = [S^D, T] = [S^*, T^D] = 0$ with $S, T \in (n+k)$ -D-operator we have; $S^2 (S^D)^{2(n+k)} = (S^*(S^D)^{n+k})^2$ and

$$T^2 (T^D)^{2(n+k)} = (T^*(T^D)^{n+k})^2, \text{ hence}$$

$$\begin{aligned} (TS)^2 ((TS)^D)^{2(n+k)} &= (TS)^*(TS)^*((TS)^D)^{n+k} ((TS)^D)^{n+k} \\ &= S^* T^* S^* T^* (T^D)^{n+k} (S^D)^{n+k} (T^D)^{n+k} (S^D)^{n+k} \\ &= S^* S^* (S^D)^{n+k} (S^D)^{n+k} T^* T^* (T^D)^{n+k} (T^D)^{n+k} \\ &= S^2 T^2 (S^D)^{2(n+k)} (T^D)^{2(n+k)} \\ &= S^* S^* T^* T^* (S^D)^{n+k} (T^D)^{n+k} (S^D)^{n+k} (T^D)^{n+k} \\ &= S^* T^* S^* T^* (S^D)^{n+k} (T^D)^{n+k} (S^D)^{n+k} (T^D)^{n+k} \\ &= ((TS)^*(TS)^*((TS)^D)^{n+k})^2. \end{aligned}$$

Hence $TS \in (n+k)$ -D-operator.

Proposition 5. Let $S, T \in (n+k)$ -D-operator. If $TS = ST = 0$, then $S+T \in (n+k)$ -D-operator.

Proof.

$S, T \in (n+k)$ -D-operator implies; $S^2 (S^D)^{2(n+k)} = (S^*(S^D)^{n+k})^2$ and

$$T^2 (T^D)^{2(n+k)} = (T^*(T^D)^{n+k})^2.$$

$TS = ST = 0$ implies $T^* S^* = S^* T^*$ which further implies

$$\begin{aligned} ((S+T)^D)^{n+k} &= (SD)^{n+k} + (T^D)^{n+k}. \text{ Thus,} \\ (S+T)^2 ((S+T)^D)^{2(n+k)} &= (S+T)^* (S+T)^* ((S+T)^D)^{n+k} ((S+T)^D)^{n+k} \end{aligned}$$

$$\begin{aligned} &= (S^* + T^*) (S^* + T^*) (S^D + T^D)^{n+k} (S^D + T^D)^{n+k} \\ &= (S^2 + T^2) ((S^D)^{n+k} + (T^D)^{n+k}) \\ &= S^2 (S^D)^{2(n+k)} + T^2 (T^D)^{2(n+k)} \\ &= (S^*(S^D)^{n+k})^2 + (T^*(T^D)^{n+k})^2 \\ &= (S^*(S^D)^{n+k} + T^*(T^D)^{n+k}) (S^*(S^D)^{n+k} + T^*(T^D)^{n+k}) \\ &= (S^* + T^*) ((S^D)^{n+k} + (T^D)^{n+k}) (S^* + T^*) ((S^D)^{n+k} + (T^D)^{n+k}) \\ &= ((S+T)^* ((S+T)^D)^{n+k})^2. \end{aligned}$$

Hence $S+T \in (n+k)$ -D-operator.

Theorem 6. Let $T_{a1}, T_{a2}, \dots, T_{aq} \in (n+k)$ -D-operator.

, then it follows that;

- i. $T_{a1} \oplus T_{a2} \oplus \dots \oplus T_{aq} \in (n+k)$ -D-operator.
- ii. $T_{a1} \otimes T_{a2} \otimes \dots \otimes T_{aq} \in (n+k)$ -D-operator.

Proof. (i) . $T_{aj} \in (n+k)$ -D-operator for all $aj = 1, 2, \dots, aq$ implies;

$$\begin{aligned} T_{aj}^2 (T_{aj}^D)^{2(n+k)} &= (T_{aj}^* (T_{aj}^D)^{n+k})^2 \text{ thus} \\ (T_{a1} \oplus T_{a2} \oplus \dots \oplus T_{aj})^2 ((T_{a1} \oplus T_{a2} \oplus \dots \oplus T_{aj})^D)^{2(n+k)} \\ &= T_{a1}^2 (T_{a1}^D)^{2(n+k)} \oplus T_{a2}^2 (T_{a2}^D)^{2(n+k)} \oplus \dots \oplus T_{aj}^2 (T_{aj}^D)^{2(n+k)} \\ &= (T_{a1}^* (T_{a1}^D)^{n+k})^2 \oplus (T_{a2}^* (T_{a2}^D)^{n+k})^2 \oplus \dots \oplus (T_{aj}^* (T_{aj}^D)^{n+k})^2 \end{aligned}$$

$$\begin{aligned}
 & T_{\alpha_1}^{(D)n+k} \\
 &= T_{\alpha_1}^* (T_{\alpha_1}^{(D)n+k} T_{\alpha_1}^* (T_{\alpha_1}^{(D)n+k} \oplus T_{\alpha_2}^* (T_{\alpha_2}^{(D)n+k} T_{\alpha_2}^* \\
 & (T_{\alpha_2}^{(D)n+k} \oplus \dots \oplus T_{\alpha_j}^* (T_{\alpha_j}^{(D)n+k} T_{\alpha_j}^* (T_{\alpha_j}^{(D)n+k} \\
 &= T_{\alpha_1}^* (T_{\alpha_1}^{(D)n+k} \oplus T_{\alpha_2}^* (T_{\alpha_2}^{(D)n+k} \oplus \dots \oplus T_{\alpha_j}^* (T_{\alpha_j}^{(D)n+k} \\
 &= ((T_{\alpha_1}^* \oplus T_{\alpha_2}^* \oplus \dots \oplus T_{\alpha_j}^*) ((T_{\alpha_1}^{(D)n+k} \oplus (T_{\alpha_2}^{(D)n+k} \oplus \\
 & \dots \oplus (T_{\alpha_j}^{(D)n+k})) \\
 &= ((T_{\alpha_1} \oplus T_{\alpha_2} \oplus \dots \oplus T_{\alpha_j})^* ((T_{\alpha_1} \oplus T_{\alpha_2} \oplus \dots \oplus \\
 & T_{\alpha_j})^{(D)n+k})^2
 \end{aligned}$$

iii. The proof for (ii) follows similarly.

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