On Extension Of D-Operator Operators

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Abstract- In this paper, we generalize the class of D-Operator by extending this study to n-D-Operator and investigate the basic properties of this class. We also show that this class is closed under strong operator topology.

Indexed Terms- D-Operator, Class (Q), Almost Class (Q), (α, β)-Class (Q), Normal operators, n-Normal, n-D-Operator operators.

I. INTRODUCTION

Throughout this paper, H is a separable complex Hilbert space, B(H) is the Banach algebra of all bounded linear operators. n-normal if $T^* T^n = T^n T^*$, $T \in B(H)$ is normal if $T^*T = TT^*$, quasinormal if $T(T^*T) = (T^*T)T$. D-Operator if $T^{*2}(T^D)^2 = (T^*T^D)^2$ (1), class (Q) if $T^{*2}T^2 = (T^*T)^2$ (5), n-power class (Q) if $T^{*2}(T^n)^2 = (T^*T^n)^2$ (6), n-D-Operator if $T^{*2}(T^D)^{2n} = (T^*(T^D)^n)^2$, for any positive integer n. We note that n-D-Operator is D-Operator when n=1.

II. MAIN RESULTS

Definition 1. Let $T \in B(H)$ be Drazin invertible. Then an operator T is called nD-Operator, denoted by, [nD], if $T^{*2}(T^D)^{2n} = (T^*(T^D)^n)^2$, for any positive integer n.

Proposition 2. Let $T \in [nD]$, then the following holds;

i. $\lambda T \in [nD]$ for every scalar λ.
ii. $S \in [nD]$ for every $S \in B(H)$ that is unitarily equivalent to T.
iii. The restriction/M of T to any closed subspace M of H which reduces T is in [nD].
iv. $(T^D)^n \in [nD]$.

Proof.

i. The proof is trivial.
ii. Since S is unitarily equivalent to T, there exists a unitary operator $U \in B(H)$ such that $S=UTU^*$. Hence;

$S^{*2n}(S^D)^{2n} = (UT^*U^*)2(U(T^D)^{nU^*})^2$
$= (UT^*U^*)(UT^*U^*)(U(T D)^nU^*)(U(T^D)^nU^*)$
$= UT^*T^*(T^D)^n(T^D)^nU^*$
$= UT^{*2}(T^D)^{2n}U^*$
$= U(T^*(T^D)^n)^2U^*$
$= UT^*(T^D)^nT^*(T^D)^nU^*$
$= (UT^*U^*)(U(T^D)^nU^*)(UT^*U^*)(U(T^D)^nU^*)$
$= S^*(S^D)^nS^*(S^D)^n$
$= (S^*(S^D)^n)^{2.}$

Thus $S \in [nD]$.

iii. $(T/M)^{*2}((T/M)\ D)^{2n} = (T/M)^*(T/M)^*((T/M)^D)^n((T/M)^D)^n$
$= (T^*/M)(T^*/M)((T^D)^n/M)((T^D)^n/M)$
$= (T^*T^*/M)((T^D)^{nT\ D})^n/M)^2$
$= (T^{*2}/M)((T^D)^{2n}/M)$
$= (T^{*2}(T^D)^{2n})/M$
$= (T^*(T^D)^nT^*(T^D)^n)/M$
$= ((T^*(T^D)^n)/M)((T^*(T^D)^n)/M)$
$= ((T^*/M)((T^D)^n/M)(T^*/M)((T^D)^n/M))$
$= ((T^*/M)((T^D)^{n/}M)^2$
$= ((T/M)^*((T/M)^D)^n)^{2.}$

Hence $T/M \in [nD]$.

iv. Suppose $T \in [nD]$, then;
$T^{*2n}(T^D)^{2n} = (T^*(T^D)^n)^2$, hence
$T^*T^*(T^D)^n(T^D)^n = T^*(T^D)^nT^*(T^D)^n$
taking adjoins on both sides

$= ((T^*)^D)^n((T^*)^D)^nTT = ((T^*)^D)^nT((T^*)^D)^nT$.
Thus $(((T^D)^n)^*)^2T^2 = (((T^D)^n)^{*)}T)^2$.

hence (T D) n ∈ [nD].

Proposition 3. The set of all n-D-Operators is a closed subset of B(H) on H.

Proof.
Let $\{T_q\}$ be a sequence of [nD] operators with $T_q \longrightarrow T$. We have to show that
$T \in [nD]$. Now $T_q \to T$ implies $T_q^* \to T^*$ and $(T_q^D)^n \to$

$(T^D)^{n.}$ Thus $T_q{}^*(T_q^D)^n \to T^*(T^D)^n$ gives $(T_q{}^*(T_q^D)^n)^2 \to (T^*(T^D)^n)^2$ …………. (0.1)

Similarly,
$T_q{}^{*2} \to T^{*2}$ and $(T_q^D)^{2n} \to (T^D)^{2n}$, thus

$T_q{}^{*2}(T_q^D)^{2n} \to T^{*2}(T^D)^{2n}$ ……………. (0.2)
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hence from (0.1) and (0.2) we have;

$\| T^{*2}(T^D)^{2n} - (T^*(T^D)^n)^2 \|$

$= \| T^{*2}(T^D)^{2n} - T_q{}^{*2}(T_q^D)^{2n} + T_q{}^{*2}(T_q^D)^{2n} - (T^*(T^D)^n)^2 \|$

$\le \| T^{*2}(T^D)^{2n} - T_q{}^{*2}(T_q^D)^{2n} \| + \| T_q{}^{*2}(T_q^D)^{2n} - (T^*(T^D)^n)^2 \|$

$= \| T^{*2}(T^D)^{2n} - T_q{}^{*2}(T_q^D)^{2n} \| + \| T_q{}^{*2}((T_q^D)^n)^2 - (T^*(T^D)^n)^2 \| \to 0$ as $q \to \infty$ and thus

$T^{*2}(T^D)^{2n} = (T^*(T^D)^n)^2$ hence $T \in [nD]$.

Proposition 4. Let $S, T \in [nD]$. If $[S, T] = [S, T^*] = 0$, then $TS \in [nD]$.

Proof
$[S, T] = [S, T^*] = 0$ implies;

$[S, T] = [S^D, T] = [S^*, T^D] = 0$ with $S, T \in [nD]$ we have; $S^{*2}(S^D)^{2n} = (S^*(S^D)^n)^2$ and
$T^{*2}(T^D)^{2n} = (T^*(T^D)^n)^2$, hence

$(TS)^{*2}((TS)^D)^{2n} = (TS)^*(TS)^*(TS)^D(TS)^D$
$= S^*T^*S^*T^*(T^D)^n(S^D)^n(T^D)^n(S^D)^n$
$= S^*S^*(S^D)^n(S^D)^nT^*T^*(T^D)^n(T^D)^n$
$= S^{*2}T^{*2}(S^D)^{2n}(T^D)^{2n}$
$= S^*S^*T^*T^*(S^D)^n(T^D)^n(S^D)^n(T^D)^n$
$= S^*T^*S^*T^*(S^D)^n(T^D)^n(S^D)^n(T^D)^n$
$= (TS)^*(TS)^*((TS)^D)^n)^2$.

Hence $TS \in [nD]$.

Proposition 5. Let $S, T \in [nD]$. If $TS = ST = 0$, then $S+T \in [nD]$.

Proof.
$S, T \in [nD]$ implies; $S^{*2}(S^D)^{2n} = (S^*(S^D)^n)^2$ and
$T^{*2}(T^D)^{2n} = (T^*(T^D)^n)^2$.
TS=ST=0 implies $T^*S^* = S^*T^*$ which further implies $((S+T)^D)^n = (SD)^n + (T^D)^{n.}$ Thus,
$= (S+T)^{*2}((S+T)D)^{2n} = (S+T)^*(S+T)^*((S+T)D)^n((S+T)D)^n$

$= (S^* + T^*)(S^* + T^*)(S^D + T^D)^n(S^D + T^D)^n$
$= (S^{*2} + T^{*2})((S^D)^{2n} + (T^D)^{2n})$
$= S^{*2}(S^D)^{2n} + T^{*2}(T^D)^{2n}$
$= (S^*(S^D)^n)^2 + (T^*(T^D)^n)^2$
$= (S^*(S^D)^n + T^*(T^D)^n)(S^*(S^D)^n + T^*(T^D)^n)$
$= (S^* + T^*)((S^D)^n + (T^D)^n)(S^* + T^*)((S^D)^n + (T^D)^n)$
$= ((S+T)^*((S+T)^D)^n)^2$.

Hence $S+T \in [nD]$.

Theorem 6. Let $T_{\alpha 1}, T_{\alpha 2}.........T_{\alpha q} \in [nD]$, then it follows that;
i. $T_{\alpha 1} \oplus T_{\alpha 2} \oplus ...... \oplus T_{\alpha q} \in [nD]$.
ii. $T_{\alpha 1} \otimes T_{\alpha 2} \otimes ...... \otimes T_{\alpha q} \in [nD]$.

Proof.
$T_{\alpha j} \in [nD]$ for all $\alpha j = 1, 2, ...\alpha q$ implies;
$T_{\alpha j}{}^{*2}(T_{\alpha j}{}^D)^{2n} = (T_{\alpha j} * (T_{\alpha j}{}^D)^n)^2$ thus
$(T_{\alpha 1} \oplus T_{\alpha 2} \oplus ...... \oplus T_{\alpha j})^{*2}((T_{\alpha 1} \oplus T_{\alpha 2} \oplus ...... \oplus T_{\alpha j})^D)^{2n}$
$= T_{\alpha 1}{}^{*2}(T_{\alpha 1}{}^D)^{2n} \oplus T_{\alpha 2}{}^{*2}(T_{\alpha 2}{}^D)^{2n} \oplus ...... \oplus T_{\alpha j}{}^{*2}(T_{\alpha j}{}^D)^{2n}$
$= (T_{\alpha 1}{}^*(T_{\alpha 1}{}^D)^n)^2 \oplus (T_{\alpha 2}{}^*(T_{\alpha 2}{}^D)^n)^2 \oplus ...... \oplus (T_{\alpha j}{}^*(T_{\alpha j}{}^D)^n)^2$
$= T_{\alpha 1}{}^*(T_{\alpha 1}^D)^nT_{\alpha 1}{}^*(T_{\alpha 1}{}^D)^n \oplus T_{\alpha 2}{}^*(T_{\alpha 2}^D)^nT_{\alpha 2}{}^*(T_{\alpha 2}{}^D)^n \oplus .... \oplus T_{\alpha j}{}^*(T_{\alpha j}{}^D)^nT_{\alpha j}{}^*(T_{\alpha j}{}^D)^n$
$= T_{\alpha 1}{}^*(T_{\alpha 1}{}^D)^n \oplus T_{\alpha 2}{}^*(T_{\alpha 2}{}^D)^n \oplus .... \oplus T_{\alpha j}{}^*(T_{\alpha j}{}^D)^n$
$= ((T_{\alpha 1}{}^* \oplus T_{\alpha 2}{}^* \oplus ...... \oplus T_{\alpha j}{}^*)((T_{\alpha 1}{}^D)^n \oplus (T_{\alpha 2}{}^D)^n \oplus ...... \oplus (T_{\alpha j}{}^D)^n))$
$= ((T_{\alpha 1} \oplus T_{\alpha 2} \oplus ...... \oplus T_{\alpha j})^*((T_{\alpha 1} \oplus T_{\alpha 2} \oplus ...... \oplus T_{\alpha j})^D)^n)^2$

(ii) . The proof for (ii) follows similarly.

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