On class (BQ) Operators of order n.

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***Abstract-** In this paper, we extend the class of (BQ) operators acting on a complex Hilbert space H to the class of (BQ) of order n. An operator if $T \in B(H)$ is said to belong to class (BQ) of order n if $T^{*2n}T^2$ commutes with $(T^{*n}T)^2$ that is $[T^{*2n}T^2, (T^{*n}T)^2] = 0$. We examine properties that this class is honored to have. We examine the relation of this class to that of class (Q) order n.*

***Indexed Terms-** Class (BQ) of order n, Class (BQ) operator, Normal operator of order n.*

I. INTRODUCTION

Throughout this paper , H denotes the usual Hilbert space over the complex field and B(H) the Banach algebra of all bounded linear algebra on an infinite dimensional separable Hilbert space H . An operator $T \in B(H)$ is said to be class (Q) if $T^{*2}T^2 = (T^*T)^2$ (1), class (Q) of order n if $T^{*2n}T^2 = (T^{*n}T)^2$, class (BQ) if $T^{*2}T^2(T^*T)^2 = (T^*T)^2T^{*2}T^2$ (5). A conjugation on a Hilbert space H is an anti-linear operator C from Hilbert space H onto itself that satisfies

Cx, Cyi = hx, yi for every $x, y \in H$ and $C^2 = I$. An operator T is said to be complex symmetric if $T = CT^*C$.

II. MAIN RESULTS

Theorem 1. Let $T \in B(H)$ be such that it's a (BQ) of order n, then the following are also equivalent;
(i). λT for any real λ
(ii). Any $S \in B(H)$ that is unitarily equivalent to T.
(iii). The restriction T-M to any closed subspace M of H.

Proof. (i). The proof is trivial.
(ii). Let $S \in B(H)$ be unitarily equivalent to T, then there exists a unitary operator $U \in B(H)$ with $S = U^*TU$ and $S^* = U^*T^{*n}U$. Since $T \in B(Q)$ of order n, we have; $T^{*2n}T^2(T^{*n}T)^2 = (T^{*n}T)^2T^{*2n}T^2$, hence
$S^{*2n}S^2(S^{*n}S)^2 = UT^{*2n}U^*UT^2U^*(UT^{*n}U^*UTU^*)^2$
$= UT^{*2n}U^*U^*T^2U^*UT^{*n}U^*UT^{*n}U^*UTU^*UTU^*$
$= UT^{*2n}T^2(T^{*n}T)^2U^*$
$= U(T^{*n}T)^2T^{*2n}T^2U^*$
and
$(S^{*n}S)^2S^{*2n}S^2 = (UT^{*n}U^*UTU^*)^2UT^{*2n}U^*UT^2U^*$
$= UT^{*n}U^*UTU^*UT^{*n}U^*UTU^*UT^{*2n}U^*UT^2U^*$
$= UT^{*n}TT^{*n}TT^{*2n}T^{*2n}U^*$
$= U(T^{*n}T)^2T^{*2n}T^2U^*$
Thus S is unitarily equivalent to T.

(iii) . If T is in class (BQ) of order n , then; $T^{*2n}T^2(T^{*n}T)^2 = (T^{*n}T)^2T^{*2n}T^2$.
Hence;
$(T/M)^{*2n}(T/M)^2\{(T/M)^{*n}(T/M)\}^2$
$=(T/M)^2(T/M)^2\{(T/M)^{*n}(T/M)\}^2$
$= (T^{*2n}/M)(T^2/M)\{(T^{*n}/M)(T/M)\}\{(T^{*n}/M)(T/M)\}$
$= \{(T^{*n}T)^2/M\}\{T^{*2n}T^2/M\}$
$= \{(T^{*n}/M)(T/M)\}^2(T/M)^{*2n}(T/M)^2$
Thus $T/M \in$ (BQ) of order n .

Theorem 2. If $T \in B(H)$ is in Class (Q) of order n, then $T \in$ (BQ) of order n .

Proof. If $T \in$ (Q) of order n, then $T^{*2n}T^2 = (T^{*n}T)^2$ post multiplying both sides by $T^{*2n}T^2$;
$T^{*2n}T^2T^{*2n}T^2 = (T^{*n}T)^2T^{*2n}T^2$
$T^{*2n}T^2T^{*n}TT^{*n}T = (T^{*n}T)^2T^{*2n}T^2$
$T^{*2n}T^2(T^{*n}T)^2 = (T^{*n}T)^2T^{*2n}T^2$.

Theorem 3. Let $S, T \in$ (BQ) of order n. If both S and T are doubly commuting, then ST is in (BQ) of order n.

Proof.
$(ST)^{*2n}(ST)^2((ST)^{*n}(ST))^2$
$=S^{*2n}T^{*2n}S^2T^2((ST)^{*n}(ST))((ST)^{*n}(ST))$
$=S^{*2n}T^{*2n}S^2T^2((S^{*n}T^{*n})(ST))((S^{*n}T^{*n})(ST))$
$=S^{*2n}T^{*2n}S^2T^2S^{*n}T^{*n}STS^{*n}T^{*n}STS^{*n}T^{*n}ST$
$= S^{*2n}T^{*2n}S^2T^2S^{*n}ST^{*n}TS^{*n}ST^{*n}T$
$=T^{*2n}T^2S^{*2n}S^2S^{*n}SS^{*n}ST^{*n}TT^{*n}T$

$=T^{*2n}T^2S^{*2n}S^2(S^{*n}S)^2T^{*n}TT^{*n}T$

$=T^{*2n} T^{2}(S^{*n}S)^{2} S^{*2n} S^{2} T^{*n} TT^{*n}T$ (Since $S \in (BQ)$ of order n).
$= (S^{*n}S)^{2} T^{*2n} T^{2} T^{*n} TT^{*n}TS^{*2n} S^{2}$
$=(S^{*n}S)^{2} T^{*2n} T^{2} (T^{*n}T)^{2} S^{*2n} S^{2}$
$=(S^{*n}S)^{2} (T^{*n}T)^{2} T^{*2n} T^{2} S^{*2n} S^{2}$ (Since $T \in (BQ)$ of order n).
$=((S^{*n}S)(T^{*n}T))^{2} T^{*2n} S^{*2n} T^{2} S^{2}$
$=((S^{*n}T^{*n})(ST))^{2} S^{*2n} T^{*2n} S^{2} T^{2}$
$= ((ST)^{*n}(ST))^{2} (ST)^{*2n} (ST)^{2}$
Thus $ST \in (BQ)$ of order n.

Theorem 4. Let $T \in B(H)$ be a class (BQ) operator of order n such that $T = CT^{*n}C$ with C being a conjugation on H. If C is such that it commutes with $T^{*2n} T^{2}$ and $(T^{*n}T)^{2}$, then T is a class (Q) operator of order n.

Proof. Let $T \in (BQ)$ of order n and complex symmetric, then we have; $T^{*2n} T^{2} (T^{*n}T)^{2} = (T^{*n}T)^{2} T^{*2n}T^{2}$ and $T = CT^{*n}C$. hence;
$T^{*2n} T^{2}(T^{*n}T)^{2} = (T^{*n}T)^{2} T^{*2n} T^{2}$
$T^{*2n}T^{2} CTCCT^{*n} CCTCCT^{*n} C = (T^{*n}T)^{2} CTCCT^{*n} CCTCCT^{*n} C$.
$T^{*2n} T^{2} CTT^{*n} TT^{*n} C = (T^{*n}T)^{2} CTT^{*n}TT^{*n} C$
$T^{*2n} T^{2} CT^{2} T^{*2n} C = (T^{*n}T)^{2} CT^{*n}TT^{*n}TC$
$T^{*2n}T^{2} CT^{*2n} T^{2} C = (T^{*n}T)^{2} C(T^{*n}T)^{2} C$.
C commutes with $T^{*2} T^{2}$ and $(T^{*}T)^{2}$ thus we get ;
$T^{*2n}T^{2}T^{*2n}T^{2} = (T^{*n}T)^{2} (T^{*n}T)^{2}$.
which implies ;
$T^{*2n}T^{2} = (T^{*n}T)^{2}$ and hence $T \in (Q)$ of order n.

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