

Coefficient Estimates for a Combined Subclass of Classes of Convex Functions and Starlike Functions

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Abstract- We introduce a class of analytic functions and obtain sharp upper bounds of the functional $|a_3 - \mu a_2^2|$ for the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ belonging to this class with special character that it tends to the class of Starlike functions as $\alpha \rightarrow \frac{\pi}{2}$.

Indexed Terms- Univalent functions, Starlike functions, Close to convex functions and bounded functions.

I. INTRODUCTION

Let \mathcal{A} denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $\mathbb{E} = \{z: |z| < 1\}$. Let \mathcal{S} be the class of functions of the form (1.1), which are analytic univalent in \mathbb{E} .

In 1916, Bieber Bach ([1], [2]) proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. In 1923, Löwner proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$.

With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was expected to try to find some relation between a_3 and a_2^2 for the class \mathcal{S} , Fekete and Szegö [5] used Löwner's method to prove the following well known result for the class \mathcal{S} .

Let $f(z) \in \mathcal{S}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if } \mu \leq 0; \\ 1 + 2 \exp\left(\frac{-2\mu}{1-\mu}\right), & \text{if } 0 \leq \mu \leq 1; \\ 4\mu - 3, & \text{if } \mu \geq 1. \end{cases} \tag{1.2}$$

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes \mathcal{S} ([7], [8], [18]-[43]).

Let us define some subclasses of \mathcal{S} .

We denote by \mathcal{S}^* , the class of univalent starlike functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$$

and satisfying the condition

$$\operatorname{Re} \left(\frac{zg'(z)}{g(z)} \right) > 0, z \in \mathbb{E}. \tag{1.3}$$

We denote by \mathcal{K} , the class of univalent convex functions

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z \in \mathcal{A}$$

and satisfying the condition

$$\operatorname{Re} \left(\frac{zh'(z)}{h(z)} \right) > 0, z \in \mathbb{E}. \tag{1.4}$$

A function $f(z) \in \mathcal{A}$ is said to be close to convex if there exists $g(z) \in \mathcal{S}^*$ such that

$$\operatorname{Re} \left(\frac{zf'(z)}{g(z)} \right) > 0, z \in \mathbb{E}. \tag{1.5}$$

The class of close to convex functions is denoted by \mathcal{C} and was introduced by Kaplan [7] and it was shown by him that all close to convex functions are univalent.

$$\mathcal{S}^*(A, B) = \left\{ f(z) \in \mathcal{A}; \frac{zf'(z)}{f(z)} < \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E} \right\} \tag{1.6}$$

$$\mathcal{K}(A, B) = \left\{ f(z) \in \mathcal{A}; \frac{(zf'(z))'}{f'(z)} < \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E} \right. \quad (1.7)$$

$$\mathcal{KS}^*(A, B, \alpha, \beta, \gamma) = \left\{ f(z) \in \mathcal{A}; \tan \alpha \left(\frac{zf'(z)}{f(z)} \right)^\beta + zf'z'f'z1-\beta < (1+\tan \alpha)1+Az1+Bz\gamma; z \in \mathbb{E} \right. \quad (1.9)$$

It is obvious that $S^*(A, B)$ is a subclass of S^* and $\mathcal{K}(A, B)$ is a subclass of \mathcal{K} .

We introduce a new subclass as

$$\left\{ f(z) \in \mathcal{A}; \tan \alpha \left(\frac{zf'(z)}{f(z)} \right)^\beta + \left(\frac{zf'(z)}{f'(z)} \right)^{1-\beta} < (1 + \tan \alpha)1 + Aw(z)1 - Bw(z); z \in \mathbb{E} \right.$$

and we will denote this class as $\mathcal{KS}^*(\alpha, \beta)$.

We will deal with two subclasses of $\mathcal{KS}^*(\alpha, \beta)$ defined as follows in our next paper:

$$\mathcal{KS}^*(\alpha, \beta, A, B) = \left\{ f(z) \in \mathcal{A}; \tan \alpha \left(\frac{zf'(z)}{f(z)} \right)^\beta + zf'z'f'z1-\beta < (1+\tan \alpha)1+Az1+Bz; z \in \mathbb{E} \right. \quad (1.8)$$

Symbol $<$ stands for subordination, which we define as follows:

- Principle of Subordination: Let $f(z)$ and $F(z)$ be two functions analytic in \mathbb{E} . Then $f(z)$ is called subordinate to $F(z)$ in \mathbb{E} if there exists a function $w(z)$ analytic in \mathbb{E} satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$; $z \in \mathbb{E}$ and we write $f(z) < F(z)$.

By \mathcal{U} , we denote the class of analytic bounded functions of the form

$$w(z) = \sum_{n=1}^{\infty} d_n z^n, w(0) = 0, |w(z)| < 1. \quad (1.10)$$

$$\text{It is known that } |d_1| \leq 1, |d_2| \leq 1 - |d_1|^2. \quad (1.11)$$

II. PRELIMINARY LEMMAS

For $0 < c < 1$, we write $w(z) = \left(\frac{c+z}{1+cz} \right)$ so that

$$\frac{1+w(z)}{1-w(z)} = 1 + 2cz + 2z^2 + \dots \quad (2.1)$$

III. MAIN RESULTS

THEOREM 3.1: Let $f(z) \in \mathcal{KS}^*(\alpha, \beta)$, then

$$\left| a_3 - \mu a_2^2 \right| \leq \begin{cases} \text{if } \mu \leq \frac{\beta + [(\beta^2 - 3\beta)\tan \alpha - 2A - 4 - 2\beta^2 + 4\beta](1 + \tan \alpha)(A - B) - (A - B)(1 + \tan \alpha)\mu}{(A - B)(1 + \tan \alpha)\{2\beta \tan \alpha + 6(1 - \beta)\}}; & (3.1) \\ \text{if } \frac{\beta + [(\beta^2 - 3\beta)\tan \alpha - 2A - 4 - 2\beta^2 + 4\beta](1 + \tan \alpha)(A - B) - \beta^2 \tan^2 \alpha - 4 - 4\beta^2 + 8\beta - 4\beta \tan \alpha + 4\beta^2 \tan \alpha}{(A - B)} \leq \mu \leq \frac{\beta + [(\beta^2 - 3\beta)\tan \alpha - 2A - 4 - 4\beta^2 + 8\beta](1 + \tan \alpha)(A - B) - \beta^2 \tan^2 \alpha - 4 - 4\beta^2 + 8\beta - 4\beta \tan \alpha + 4\beta^2 \tan \alpha}{(A - B)(1 + \tan \alpha)\{2\beta \tan \alpha + 6(1 - \beta)\}}; & (3.2) \\ \text{if } \mu \geq \frac{\mu(A - B)(1 + \tan \alpha) - \frac{\beta + [(\beta^2 - 3\beta)\tan \alpha - 2A - 4 - 2\beta^2 + 4\beta](1 + \tan \alpha)(A - B) - \beta^2 \tan^2 \alpha - 4 - 4\beta^2 + 8\beta - 4\beta \tan \alpha + 4\beta^2 \tan \alpha}{\{2\beta \tan \alpha + 6(1 - \beta)\}}}{\frac{\beta + [(\beta^2 - 3\beta)\tan \alpha - 2A - 4 - 2\beta^2 + 4\beta](1 + \tan \alpha)(A - B) - \beta^2 \tan^2 \alpha - 4 - 4\beta^2 + 8\beta - 4\beta \tan \alpha + 4\beta^2 \tan \alpha}{(A - B)(1 + \tan \alpha)\{2\beta \tan \alpha + 6(1 - \beta)\}}} & (3.3) \end{cases}$$

The results are sharp.

Proof: By definition of $\mathcal{K}\mathcal{S}^*(\alpha, \beta)$, we have

$$\tan\alpha \left(\frac{zf'(z)}{f(z)}\right)^\beta + \left(\frac{(zf'(z))'}{f'(z)}\right)^{1-\beta} = (1 + \tan\alpha) \frac{1+w(z)}{1-w(z)}; w(z) \in \mathcal{U}. \tag{3.4}$$

$$\begin{aligned} & \tan\alpha \left\{ 1 + \beta a_2 z + (2\beta a_3 + \frac{\beta(\beta-3)}{2} a_2^2) z^2 + \dots \right. \\ & \left. - +1+21-\beta a_2 z + 2(1-\beta)(3a_3 - (\beta+2)a_2^2) z^2 + \dots \right. \\ & \left. = (1+\tan\alpha)1 + A - Bc_1 z + (A-B)c_2 - Bc_1^2 z^2 + \dots \right. \end{aligned} \tag{3.5}$$

Identifying terms in (3.5), we get

$$a_2 = \frac{(A-B)(1+\tan\alpha)}{\beta \tan\alpha + 2(1-\beta)} c_1 \tag{3.6}$$

Expanding the series (3.4), we get

$$a_3 = \frac{(1+\tan\alpha)(A-B)}{2\beta \tan\alpha + 3(2-2\beta)} c_2 + \left[\frac{\beta(A-B)(1+\tan\alpha) + \beta(\beta-3)\tan\alpha - 2(A+2) + 2\beta(\beta+2)}{2\beta \tan\alpha + 6-6\beta} \left(\frac{(A-B)^2(1+\tan\alpha)^2}{(\beta \tan\alpha + 2(1-\beta))^2} \right) \right] c_1^2. \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{(1+\tan\alpha)(A-B)}{2\beta \tan\alpha + 6-6\beta} c_2 + \frac{(\beta(A-B) + \beta(\beta-3)\tan\alpha - 2(A+2) + 2\beta(\beta+2))(A-B)(1+\tan\alpha)}{2\beta \tan\alpha + 6-6\beta} c_1^2 + \frac{(A-B)(1+\tan\alpha)}{(\beta \tan\alpha + 2(1-\beta))^2} \\ & - \frac{\mu(A-B)^2(1+\tan\alpha)^2 c_1^2}{(\beta \tan\alpha + 2(1-\beta))^2}. \end{aligned} \tag{3.8}$$

Taking absolute value and using Triangular inequality, (3.8) can be rewritten as

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq 1 - |c_1|^2 \left[\frac{(1+\tan\alpha)(A-B)}{2\beta \tan\alpha + 6-6\beta} + \frac{(\beta(A-B) + \beta(\beta-3)\tan\alpha - 2(A+2) + 2\beta(\beta+2))(A-B)(1+\tan\alpha)}{2\beta \tan\alpha + 6-6\beta} c_1^2 \right] \frac{(A-B)(1+\tan\alpha)}{(\beta \tan\alpha + 2(1-\beta))^2} \\ & - \frac{\mu(A-B)^2(1+\tan\alpha)^2 c_1^2}{(\beta \tan\alpha + 2(1-\beta))^2} \end{aligned} \tag{3.9}$$

Using (1.9) in (3.9), Simple calculations yield

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{(1+\tan\alpha)(A-B)}{2\beta \tan\alpha + 6-6\beta} + \frac{(\beta(A-B) + \beta(\beta-3)\tan\alpha - 2(A+2) + 2\beta(\beta+2))(A-B)(1+\tan\alpha)}{2\beta \tan\alpha + 6-6\beta} c_1^2 \frac{(A-B)(1+\tan\alpha)}{(\beta \tan\alpha + 2(1-\beta))^2} \\ & - \frac{\mu(A-B)^2(1+\tan\alpha)^2 c_1^2}{(\beta \tan\alpha + 2(1-\beta))^2} - c_1^2 \frac{(1+\tan\alpha)(A-B)(\beta \tan\alpha + 2(1-\beta))^2}{(\beta \tan\alpha + 2(1-\beta))^2(2\beta \tan\alpha + 6-6\beta)}. \end{aligned} \tag{3.10}$$

In this case, (3.10) can be rewritten as

Case I:

$$\mu \leq \frac{\beta + [(\beta^2 - 3\beta)\tan\alpha - 2A - 4 - 2\beta^2 + 4\beta](1+\tan\alpha)(A-B)}{(1+\tan\alpha)(A-B)(2\beta \tan\alpha + 6-6\beta)}.$$

$$|a_3 - \mu a_2^2| \leq \frac{(1+\tan\alpha)(A-B)}{2\beta \tan\alpha + 3(2-2\beta)} + \frac{(1+\tan\alpha)(A-B)}{(\beta \tan\alpha + 2(1-\beta))^2} \left[\frac{\beta + [(\beta^2 - 3\beta)\tan\alpha - 2A - 4 - 2\beta^2 + 4\beta](1+\tan\alpha)(A-B)}{2\beta \tan\alpha + 6(1-\beta)} - \mu(1 + \tan\alpha) \right] |c_1|^2. \tag{3.11}$$

Subcase I (a):

$$\mu \leq \frac{\beta + [(\beta^2 - 3\beta)\tan\alpha - 2A - 4 - 2\beta^2 + 4\beta](1+\tan\alpha)(A-B) - 4(1-\beta^2 - 2\beta) - 4\beta \tan\alpha + 4\beta^2 \tan\alpha}{(A-B)(1+\tan\alpha)\{2\beta \tan\alpha + 6(1-\beta)\}}. \tag{3.11}$$

Using (1.9), (3.11) becomes

$$|a_3 - \mu a_2^2| \leq \frac{(1+\tan\alpha)(A-B)}{(\beta \tan\alpha + 2(1-\beta))^2} \left[\frac{\beta + [(\beta^2 - 3\beta)\tan\alpha - 2A - 4 - 2\beta^2 + 4\beta](1+\tan\alpha)(A-B)}{2\beta \tan\alpha + 6(1-\beta)} - (A-B)(1 + \tan\alpha)\mu \right] \tag{3.12}$$

Subcase I (b):

$$\mu \geq \frac{\beta + [(\beta^2 - 3\beta)\tan\alpha - 2A - 4 - 2\beta^2 + 4\beta](1 + \tan\alpha)(A - B) - \beta^2 \tan^2 \alpha - 4 - 4\beta^2 + 8\beta - 4\beta \tan\alpha + 4\beta^2 \tan\alpha}{(A - B)(1 + \tan\alpha)\{2\beta \tan\alpha + 6(1 - \beta)\}}$$

We obtain from (3.11)

$$|a_3 - \mu a_2^2| \leq \frac{(1 + \tan\alpha)(A - B)}{2\beta \tan\alpha + 6(1 - \beta)} \tag{3.13}$$

Case II:

$$\mu \geq \frac{\beta + [(\beta^2 - 3\beta)\tan\alpha - 2A - 4 - 2\beta^2 + 4\beta](1 + \tan\alpha)(A - B)}{4(1 + \tan\alpha)(\beta \tan\alpha + 3(1 - \beta))}$$

Preceding as in case I, we get

$$|a_3 - \mu a_2^2| \leq \frac{(1 + \tan\alpha)(A - B)}{6(1 - \beta) + 2\beta \tan\alpha} + \frac{(1 + \tan\alpha)(A - B)}{\{\tan\alpha\beta + 2(1 - \beta)\}^2} \left[\frac{\beta + [(\beta^2 - 3\beta)\tan\alpha - 2A - 4 - 2\beta^2 + 4\beta](1 + \tan\alpha)(A - B) - \beta^2 \tan^2 \alpha - 4 - 4\beta^2 + 8\beta - 4\beta \tan\alpha + 4\beta^2 \tan\alpha}{\{2\beta \tan\alpha + 6(1 - \beta)\}} - \mu(A - B)(1 + \tan\alpha) \right] |c_1|^2 \tag{3.14}$$

Subcase II (a):

$$\mu \leq \frac{\beta + [(\beta^2 - 3\beta)\tan\alpha - 2A - 4 - 4\beta^2 + 8\beta](1 + \tan\alpha)(A - B) - \beta^2 \tan^2 \alpha - 4 - 4\beta^2 + 8\beta - 4\beta \tan\alpha + 4\beta^2 \tan\alpha}{(A - B)(1 + \tan\alpha)\{2\beta \tan\alpha + 6(1 - \beta)\}}$$

(3.14) takes the form

$$|a_3 - \mu a_2^2| \leq \frac{(1 + \tan\alpha)(A - B)}{2\beta \tan\alpha + 6(1 - \beta)} \tag{3.15}$$

Combining subcase I (b) and subcase II (a), we obtain

$$|a_3 - \mu a_2^2| \leq \frac{(1 + \tan\alpha)(A - B)}{2\beta \tan\alpha + 6(1 - \beta)} \text{ if } \frac{\beta + [(\beta^2 - 3\beta)\tan\alpha - 2A - 4 - 2\beta^2 + 4\beta](1 + \tan\alpha)(A - B) - \beta^2 \tan^2 \alpha - 4 - 4\beta^2 + 8\beta - 4\beta \tan\alpha + 4\beta^2 \tan\alpha}{(A - B)(1 + \tan\alpha)\{2\beta \tan\alpha + 6(1 - \beta)\}} \leq \mu \leq \frac{\beta + [(\beta^2 - 3\beta)\tan\alpha - 2A - 4 - 4\beta^2 + 8\beta](1 + \tan\alpha)(A - B) - \beta^2 \tan^2 \alpha - 4 - 4\beta^2 + 8\beta - 4\beta \tan\alpha + 4\beta^2 \tan\alpha}{(A - B)(1 + \tan\alpha)\{2\beta \tan\alpha + 6(1 - \beta)\}} \tag{3.16}$$

Subcase II (b): $\mu \geq \frac{\beta + [(\beta^2 - 3\beta)\tan\alpha - 2A - 4 - 2\beta^2 + 4\beta](1 + \tan\alpha)(A - B) - \beta^2 \tan^2 \alpha - 4 - 4\beta^2 + 8\beta - 4\beta \tan\alpha + 4\beta^2 \tan\alpha}{(A - B)(1 + \tan\alpha)\{2\beta \tan\alpha + 6(1 - \beta)\}}$

Preceding as in subcase I (a), we get

$$|a_3 - \mu a_2^2| \leq \mu(A - B)(1 + \tan\alpha) - \frac{(1 + \tan\alpha)(A - B)}{\{\beta \tan\alpha + 2(1 - \beta)\}^2} \left\{ \frac{\beta + [(\beta^2 - 3\beta)\tan\alpha - 2A - 4 - 2\beta^2 + 4\beta](1 + \tan\alpha)(A - B) - \beta^2 \tan^2 \alpha - 4 - 4\beta^2 + 8\beta - 4\beta \tan\alpha + 4\beta^2 \tan\alpha}{\{2\beta \tan\alpha + 6(1 - \beta)\}} \right\} \tag{3.17}$$

Combining (3.12), (3.16) and (3.17), the theorem is proved.

Extremal function for (3.1) and (3.3) is defined by

$$f_1(z) = (1 + az)^b$$

Where $a = \frac{2\gamma\{\beta+3(1-\beta)\tan\alpha\}}{\{4(1-\beta)(\beta+2)\tan\alpha-\beta(\beta-3)\{\beta+2(1-\beta)\tan\alpha\}^3-2\gamma}}$

And $b = \frac{\{4(1-\beta)(\beta+2)\tan\alpha-\beta(\beta-3)\{\beta+2(1-\beta)\tan\alpha\}^3-2\gamma}{\{\beta+3(1-\beta)\tan\alpha\}\{\beta+2(1-\beta)\tan\alpha\}}$

Extremal function for (3.2) is defined by $f_2(z) = z(1 + cz^2)^d$,

Where $c = \frac{\tan\alpha}{\beta+3(1-\beta)\tan\alpha}$ and $d = \frac{\gamma}{\tan\alpha}$

Corollary 3.2: Putting $\gamma = 1, \beta = 0$ and applying limit as $\alpha \rightarrow \frac{\pi}{2}$ in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} 1 - \mu, & \text{if } \mu \leq 1; \\ \frac{1}{3} & \text{if } 1 \leq \mu \leq \frac{4}{3}; \\ \mu - 1, & \text{if } \mu \geq \frac{4}{3} \end{cases}$$

These estimates were derived by Keogh and Merkes [9] and are results for the class of univalent convex functions.

Corollary 3.3: Putting $\alpha = 0, \beta = 1, \gamma = 0$ in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if } \mu \leq \frac{1}{2}; \\ 1 & \text{if } \frac{1}{2} \leq \mu \leq 1; \\ 4\mu - 3, & \text{if } \mu \geq 1 \end{cases}$$

These estimates were derived by Keogh and Merkes [9] and are results for the class of univalent starlike functions.

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